ON SPECIAL QUADRATIC BIRATIONAL TRANSFORMATIONS WHOSE BASE LOCUS HAS DIMENSION AT MOST THREE

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ABSTRACT. We study birational transformations $\varphi:\mathbb{P}^n \dashrightarrow \overline{\varphi(\mathbb{P}^n)} \subseteq \mathbb{P}^N$ defined by linear systems of quadrics whose base locus is smooth and irreducible of dimension ≤ 3 and whose image $\overline{\varphi(\mathbb{P}^n)}$ is sufficiently regular.

CONTENTS

INTRODUCTION

In this note we continue the study of special quadratic birational transformations $\varphi:\mathbb{P}^n \dashrightarrow \mathbf{S} := \overline{\varphi(\mathbb{P}^n)} \subseteq \mathbb{P}^N$ started in [40], by reinterpreting techniques and well-known results on special Cremona transformations (see [12], [13], [14] and [24]). While in [40] we required that $\mathbf{S}$ was a hypersurface, here we allow more freedom in the choice of $\mathbf{S}$, but we only treat the case in which the dimension of the base locus $\mathfrak{B}$ is $r=\dim(\mathfrak{B})\leq 3$. In the last section, we shall also obtain partial results in the case $r=4$.

Note that for every closed subscheme $X\subset\mathbb{P}^{n-1}$ cut out by the quadrics containing it, we can consider $\mathbb{P}^{n-1}$ as a hyperplane in $\mathbb{P}^n$ and hence X as a subscheme of $\mathbb{P}^n$. So the linear system $|\mathscr{I}_{X,\mathbb{P}^n}(2)|$ of all quadrics in $\mathbb{P}^n$ containing X defines a quadratic rational map $\psi:\mathbb{P}^n\dashrightarrow\mathbb{P}^N$ ($N=h^0(\mathscr{I}_{X,\mathbb{P}^n}(2))-1=n+h^0(\mathscr{I}_{X,\mathbb{P}^{n-1}}(2))$), which is birational onto the image and whose inverse is defined by linear forms, i.e. ψ is of type $(2,1)$. Conversely, every birational transformation $\psi:\mathbb{P}^n\dashrightarrow\overline{\psi(\mathbb{P}^n)}\subseteq\mathbb{P}^N$ of type $(2,1)$ whose image is nondegenerate, normal and linearly normal arise in this way. From this it follows that there are many (special) quadratic

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transformations. However, when the image $\mathbf{S}$ of the transformation φ is sufficiently regular, by straightforward generalization of [14, Proposition 2.3], we obtain strong numerical and geometric restrictions on the base locus $\mathfrak{B}$. For example, as soon as $\mathbf{S}$ is not too much singular, the secant variety $\mathrm{Sec}(\mathfrak{B}) \subset \mathbb{P}^n$ has to be a hypersurface and $\mathfrak{B}$ has to be a *QEL*-variety of type $\delta = \delta(\mathfrak{B}) = 2\dim(\mathfrak{B}) + 2 - n$; in particular $n \leq 2\dim(\mathfrak{B}) + 2$ and $\mathrm{Sec}(\mathfrak{B})$ is a hyperplane if and only if φ is of type $(2,1)$. So the classification of transformations φ of type $(2,1)$ whose base locus has dimension ≤ 3 essentially follows from classification results on *QEL*-manifold: [37, Propositions 1.3 and 3.4], [30, Theorem 2.2] and [11, Theorems 4.10 and 7.1].

When φ is of type $(2,d)$ with $d \geq 2$, then $\mathrm{Sec}(\mathfrak{B})$ is a nonlinear hypersurface and it is not so easy to exhibit examples. The most difficult cases of this kind are those for which $n = 2r+2$ i.e. $\delta = 0$. In order to classify these transformations, we first determine the Hilbert polynomial of $\mathfrak{B}$ in Lemmas 4.2 and 5.2, by using the usual Castelnuovo's argument, Castelnuovo's bound and some refinement of Castelnuovo's bound, see [10] and [33]. Consequently we deduce Propositions 4.4 and 5.7 by applying the classification of smooth varieties of low degree: [25], [27], [29], [16], [17], [6], [26]. We also apply the double point formula in Lemmas: 4.3, 5.3, 5.4, 5.5 and 5.6, in order to obtain additional informations on d and $\Delta = \deg(\mathbf{S})$.

We summarize our classification results in Table 1. In particular, we provide an answer to a question left open in the recent preprint [4].

1. Notation and general results

Throughout the paper we work over $\mathbb{C}$ and keep the following setting.

Assumption 1.1. Let $\varphi : \mathbb{P}^n \dashrightarrow \mathbf{S} := \overline{\varphi(\mathbb{P}^n)} \subseteq \mathbb{P}^{n+a}$ be a quadratic birational transformation with smooth connected base locus $\mathfrak{B}$ and with $\mathbf{S}$ nondegenerate, linearly normal and factorial.

Recall that we can resolve the indeterminacies of φ with the diagram

$$\begin{tikzcd} & \widetilde{\mathbb{P}^n} \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ \mathbb{P}^n \arrow[rr, dashed, "\varphi"] & & \mathbf{S} \end{tikzcd} \tag{1.1}$$

where $\pi : \widetilde{\mathbb{P}^n} = \mathrm{Bl}_{\mathfrak{B}}(\mathbb{P}^n) \to \mathbb{P}^n$ is the blow-up of $\mathbb{P}^n$ along $\mathfrak{B}$ and $\pi' = \varphi \circ \pi : \widetilde{\mathbb{P}^n} \to \mathbf{S}$. Denote by $\mathfrak{B}'$ the base locus of φ^{-1}, E the exceptional divisor of π, $E' = \pi'^{-1}(\mathfrak{B}')$, $H = \pi^*(H_{\mathbb{P}^n})$, $H' = \pi'^*(H_{\mathbf{S}})$, and note that, since $\pi'|_{\widetilde{\mathbb{P}^n}\setminus E'} : \widetilde{\mathbb{P}^n} \setminus E' \to \mathbf{S} \setminus \mathfrak{B}'$ is an isomorphism, we have $(\mathrm{sing}(\mathbf{S}))_{\mathrm{red}} \subseteq (\mathfrak{B}')_{\mathrm{red}}$. We also put $r = \dim(\mathfrak{B})$, $r' = \dim(\mathfrak{B}')$, $\lambda = \deg(\mathfrak{B})$, $g = g(\mathfrak{B})$ the sectional genus of $\mathfrak{B}$, $c_j = c_j(\mathscr{T}_{\mathfrak{B}}) \cdot H_{\mathfrak{B}}^{r-j}$ (resp. $s_j = s_j(\mathscr{N}_{\mathfrak{B},\mathbb{P}^n}) \cdot H_{\mathfrak{B}}^{r-j}$) the degree of the j-th Chern class (resp. Segre class) of $\mathfrak{B}$, $\Delta = \deg(\mathbf{S})$, $c = c(\mathbf{S})$ the *coindex* of $\mathbf{S}$ (the last of which is defined by $-K_{\mathrm{reg}(\mathbf{S})} \sim (n+1-c)H_{\mathrm{reg}(\mathbf{S})}$, whenever $\mathrm{Pic}(\mathbf{S}) = \mathbb{Z}\langle H_{\mathbf{S}}\rangle$).

Assumption 1.2. We suppose that there exists a rational map $\widehat{\varphi} : \mathbb{P}^{n+a} \dashrightarrow \mathbb{P}^n$ defined by a sublinear system of $|\mathscr{O}_{\mathbb{P}^{n+a}}(d)|$ and having base locus $\widehat{\mathfrak{B}}$ such that $\varphi^{-1} = \widehat{\varphi}|_{\mathbf{S}}$ and $\mathfrak{B}' = \widehat{\mathfrak{B}} \cap \mathbf{S}$. We then will say that φ^{-1} is *liftable* and that φ is *of type* $(2,d)$.

The above assumption yields the relations:

$$\begin{array}{ll} H' \sim 2H - E, & H \sim dH' - E', \\ E' \sim (2d-1)H - dE, & E \sim (2d-1)H' - 2E', \end{array} \tag{1.2}$$

and hence also $\mathrm{Pic}(\widetilde{\mathbb{P}^n}) \simeq \mathbb{Z}\langle H\rangle \oplus \mathbb{Z}\langle E\rangle \simeq \mathbb{Z}\langle H'\rangle \oplus \mathbb{Z}\langle E'\rangle$. Note that, by the proofs of [14, Proposition 1.3 and 2.1(a)] and by factoriality of $\mathbf{S}$, we obtain that E' is a reduced and irreducible divisor. Moreover we have $\mathrm{Pic}(\mathbf{S}) \simeq \mathrm{Pic}(\mathbf{S}\setminus\mathfrak{B}') \simeq \mathrm{Pic}(\widetilde{\mathbb{P}^n}\setminus E') \simeq \mathbb{Z}\langle H'\rangle \simeq \mathbb{Z}\langle H_{\mathbf{S}}\rangle$. Finally, we require the following:[1]

Assumption 1.3. $(\mathrm{sing}(\mathbf{S}))_{\mathrm{red}} \neq (\mathfrak{B}')_{\mathrm{red}}$.

Now we point out that, since E' is irreducible, by Assumption 1.3 and [14, Theorem 1.1], we deduce that $\pi'|_V : V \to U$ coincides with the blow-up of U along Z, where $U = \mathrm{reg}(\mathbf{S})\setminus\mathrm{sing}((\mathfrak{B}')_{\mathrm{red}})$, $V = \pi'^{-1}(U)$ and $Z = U\cap(\mathfrak{B}')_{\mathrm{red}}$. It follows that $K_{\widetilde{\mathbb{P}^n}} \sim (-n-1)H + (n-r-1)E \sim (c-n-1)H' + (n-r'-1)E'$, from which, together with (1.2), we obtain $2r+3-n = n-r'-1$ and $c = (1-2d)r + dn - 3d + 2$. One can also easily see that, for the general point $x \in \mathrm{Sec}(\mathfrak{B})\setminus\mathfrak{B}$, $\overline{\varphi^{-1}(\varphi(x))}$ is a linear space of dimension $n-r'-1$ and $\overline{\varphi^{-1}(\varphi(x))}\cap\mathfrak{B}$ is a quadric hypersurface, which coincides with the entry locus $\Sigma_x(\mathfrak{B})$ of $\mathfrak{B}$ with respect to x. For more details we refer the reader to [14, Proposition 2.3] and [40, Proposition 3.1]. So we can establish one of the main results useful for our purposes:

Proposition 1.4. $\mathrm{Sec}(\mathfrak{B}) \subset \mathbb{P}^n$ *is a hypersurface of degree* $2d-1$ *and* $\mathfrak{B}$ *is a QEL-variety of type* $\delta = 2r+2-n$.

In many cases, $\mathfrak{B}$ has a much stronger property of being *QEL*-variety. Recall that a subscheme $X \subset \mathbb{P}^n$ is said to have the K_2 property if X is cut out by quadratic forms $F_0,\dots,F_N$ such that the Koszul relations among the F_i are generated by linear syzygies. We have the following fact (see [41] and [1]):

Fact 1.5. *Let* $X \subset \mathbb{P}^n$ *be a smooth variety cut out by quadratic forms* $F_0,\dots,F_N$ *satisfying* K_2 *property and let* $F = [F_0,\dots,F_N] : \mathbb{P}^n \dashrightarrow \mathbb{P}^N$ *be the induced rational map. Then for every* $x \in \mathbb{P}^n\setminus X$, $\overline{F^{-1}(F(x))}$ *is a linear space of dimension* $n+1-\mathrm{rank}\left(\left(\partial F_i/\partial x_j(x)\right)_{i,j}\right)$; *moreover,* $\dim(\overline{F^{-1}(F(x))}) > 0$ *if and only if* $x \in \mathrm{Sec}(X)\setminus X$ *and in this case* $\overline{F^{-1}(F(x))}\cap X$ *is a quadric hypersurface, which coincides with the entry locus* $\Sigma_x(X)$ *of* X *with respect to* x.

We have a simple sufficient condition for the K_2 property (see [3, Proposition 2]):

Fact 1.6. *Let* $X \subset \mathbb{P}^n$ *be a smooth linearly normal variety and suppose* $h^1(\mathscr{O}_X) = 0$ *if* $\dim(X) \geq 2$. *Putting* $\lambda = \deg(X)$ *and* $s = \mathrm{codim}_{\mathbb{P}^n}(X)$ *we have:*

- *if* $\lambda \leq 2s+1$, *then* X *is arithmetically Cohen-Macaulay;*
- *if* $\lambda \leq 2s$, *then the homogeneous ideal of* X *is generated by quadratic forms;*
- *if* $\lambda \leq 2s-1$, *then the syzygies of the generators of the homogeneous ideal of* X *are generated by the linear ones.*

Remark 1.7. Let $\psi : \mathbb{P}^n \dashrightarrow \mathbf{Z} := \overline{\psi(\mathbb{P}^n)} \subseteq \mathbb{P}^{n+a}$ be a birational transformation ($n \geq 3$).

We point out that, from Grothendieck's Theorem on parafactoriality (Samuel's Conjecture) [21, XI Corollaire 3.14] it follows that $\mathbf{Z}$ is factorial whenever it is a local complete intersection with $\dim(\mathrm{sing}(\mathbf{Z})) < \dim(\mathbf{Z}) - 3$. Of course, every complete intersection in a smooth variety is a local complete intersections.

[1] See Example 6.4 and [40, Example 4.6] for explicit examples of special quadratic birational transformations for which Assumption 1.3 is not satisfied.

Moreover, ψ^{-1} is liftable whenever $\mathrm{Pic}(\mathbf{Z}) = \mathbb{Z}\langle H_{\mathbf{Z}}\rangle$ and $\mathbf{Z}$ is factorial and projectively normal. So, from [32] and [22, IV Corollary 3.2], ψ^{-1} is liftable whenever $\mathbf{Z}$ is either smooth and projectively normal with $n \geq a+2$ or a factorial complete intersection.

2. Numerical restrictions

Proposition 1.4 already provides a restriction on the invariants of the transformation φ; here we give further restrictions of this kind.

Proposition 2.1. *Let $\varepsilon = 0$ if $\langle \mathfrak{B}\rangle = \mathbb{P}^n$ and let $\varepsilon = 1$ otherwise.*

- *If $r = 1$ we have:*

$$\begin{aligned} \lambda &= (n^2 - n + 2\varepsilon - 2a - 2)/2, \\ g &= (n^2 - 3n + 4\varepsilon - 2a - 2)/2. \end{aligned}$$

- *If $r = 2$ we have:*

$$\begin{aligned} \chi(\mathscr{O}_{\mathfrak{B}}) &= (2a - n^2 + 5n + 2g - 6\varepsilon + 4)/4, \\ \lambda &= (n^2 - n + 2g + 2\varepsilon - 2a - 4)/4. \end{aligned}$$

- *If $r = 3$ we have:*

$$\chi(\mathscr{O}_{\mathfrak{B}}) = (4\lambda - n^2 + 3n - 2g - 4\varepsilon + 2a + 6)/2.$$

Proof. By Proposition 1.4 we have $h^0(\mathbb{P}^n, \mathscr{I}_{\mathfrak{B}}(1)) = \varepsilon$. Since $\mathbf{S}$ is normal and linearly normal, we have $h^0(\mathbb{P}^n, \mathscr{I}_{\mathfrak{B}}(2)) = n+1+a$ (see [40, Lemma 2.2]). Moreover, since $n \leq 2r+2$ (being $\delta \geq 0$), proceeding as in [40, Lemma 3.3] (or applying [33, Proposition 1.8]), we obtain $h^j(\mathbb{P}^n, \mathscr{I}_{\mathfrak{B}}(k)) = 0$ for every $j, k \geq 1$. So we obtain $\chi(\mathscr{O}_{\mathfrak{B}}(1)) = n+1-\varepsilon$ and $\chi(\mathscr{O}_{\mathfrak{B}}(2)) = (n+1)(n+2)/2 - (n+1+a)$. $\square$

Proposition 2.2.

- *If $r = 1$ we have:*

$$\begin{aligned} c_1 &= 2 - 2g, \\ s_1 &= (-n-1)\,\lambda - 2g + 2, \\ d &= (2\lambda - 2^n)/\left((2n-2)\,\lambda - 2^{n+1} - 4g + 4\right), \\ \Delta &= (1-n)\,\lambda + 2^n + 2g - 2. \end{aligned}$$

- *If $r = 2$ we have:*

$$\begin{aligned} c_1 &= \lambda - 2g + 2, \\ c_2 &= -\left((n^2 - 3n)\,\lambda - 2^{n+1} + (4-4g)\,n + 4g + 2\Delta - 4\right)/2, \\ s_1 &= -n\lambda - 2g + 2, \\ s_2 &= 2n\lambda + 2^n + (4g-4)\,n - \Delta, \\ d\Delta &= (2-n)\,\lambda + 2^{n-1} + 2g - 2. \end{aligned}$$

- *If $r=3$ we have:*

$$\begin{aligned}
c_1 &= 2\lambda-2g+2,\\
c_2 &= -\left(\left(n^2-5n+2\right)\lambda-2^n+(4-4g)\,n+12g+2d\Delta-12\right)/2,\\
c_3 &= \left(\left(2n^3-12n^2+22n-12\right)\lambda+92^n+n\left(-32^n+18g+6d\Delta-18\right)\right.\\
&\quad \left.+(6-6g)\,n^2-24g+(-6d-6)\,\Delta+24\right)/6,\\
s_1 &= (1-n)\,\lambda-2g+2,\\
s_2 &= ((4n-4)\,\lambda+2^n+(8g-8)\,n-8g-2d\Delta+8)/2,\\
s_3 &= \left(\left(2n^3-12n^2+10n\right)\lambda+32^n+n\left(-32^n+12g+6d\Delta-12\right)\right.\\
&\quad \left.+(12-12g)\,n^2-3\Delta\right)/3.
\end{aligned}$$

Proof. See also [12] and [13]. By [12, page 291] we see that

$$H^j\cdot E^{n-j}=\begin{cases}1, & \text{if } j=n;\\ 0, & \text{if } r+1\le j\le n-1;\\ (-1)^{n-j-1}s_{r-j}, & \text{if } j\le r.\end{cases}$$

Since $H'=2H-E$ and $H=dH'-E'$ we have

$$\Delta = H'^n=(2H-E)^n, \tag{2.1}$$

$$d\Delta = dH'^n=H'^{n-1}\cdot(dH'-E')=(2H-E)^{n-1}\cdot H. \tag{2.2}$$

From the exact sequence $0\to\mathscr{T}_{\mathfrak{B}}\to\mathscr{T}_{\mathbb{P}^n}|_{\mathfrak{B}}\to\mathscr{N}_{\mathfrak{B},\mathbb{P}^n}\to 0$ we get:

$$s_1 = -\lambda\,(n+1)+c_1, \tag{2.3}$$

$$s_2 = \lambda\binom{n+2}{2}-c_1\,(n+1)+c_2, \tag{2.4}$$

$$s_3 = -\lambda\binom{n+3}{3}+c_1\binom{n+2}{2}-c_2\,(n+1)+c_3, \tag{2.5}$$

$$\vdots$$

Moreover $c_1=-K_{\mathfrak{B}}\cdot H_{\mathfrak{B}}^{r-1}$ and it can be expressed as a function of λ and g. Thus we found $r+3$ independent equations on the $2r+5$ variables: $c_1,\ldots,c_r,s_1,\ldots,s_r,d,\Delta,\lambda,g,n$. □

Remark 2.3. Proposition 2.2 holds under less restrictive assumptions, as shown in the above proof. Here we treat the special case: let $\psi:\mathbb{P}^8\dashrightarrow\mathbf{Z}:=\overline{\psi(\mathbb{P}^8)}\subseteq\mathbb{P}^{8+a}$ be a quadratic rational map whose base locus is a smooth irreducible 3-dimensional variety X. Without any other restriction on ψ, denoting with $\pi:\mathrm{Bl}_X(\mathbb{P}^8)\to\mathbb{P}^8$ the blow-up of $\mathbb{P}^8$ along X and with $s_i(X)=s_i(\mathscr{N}_{X,\mathbb{P}^8})$, we have

$$\deg(\psi)\deg(\mathbf{Z})=(2\pi^*(H_{\mathbb{P}^8})-E_X)^8=-s_3(X)-16s_2(X)-112s_1(X)-448\deg(X)+256. \tag{2.6}$$

Moreover, if ψ is birational with liftable inverse and $\dim(\mathrm{sing}(\mathbf{Z}))\le 6$, we also have

$$d\deg(\mathbf{Z})=(2\pi^*(H_{\mathbb{P}^8})-E_X)^7\cdot\pi^*(H_{\mathbb{P}^8})=-s_2(X)-14s_1(X)-84\deg(X)+128, \tag{2.7}$$

where d denotes the degree of the linear system defining ψ^{-1}.

Proposition 2.4 is a translation of the well-known *double point formula* (see for example [35] and [31]), taking into account Proposition 1.4.

Proposition 2.4. *If* $\delta = 0$ *then*

$$2(2d-1) = \lambda^2 - \sum_{j=0}^{r} \binom{2r+1}{j} s_{r-j}(\mathscr{T}_{\mathfrak{B}}) \cdot H_{\mathfrak{B}}^{j}.$$

3. Case of dimension 1

Lemma 3.1 directly follows from Propositions 2.1 and 2.2.

Lemma 3.1. *If* $r = 1$, *then one of the following cases holds:*

(A) $n = 3$, $a = 1$, $\lambda = 2$, $g = 0$, $d = 1$, $\Delta = 2$;
(B) $n = 4$, $a = 0$, $\lambda = 5$, $g = 1$, $d = 3$, $\Delta = 1$;
(C) $n = 4$, $a = 1$, $\lambda = 4$, $g = 0$, $d = 2$, $\Delta = 2$;
(D) $n = 4$, $a = 2$, $\lambda = 4$, $g = 1$, $d = 1$, $\Delta = 4$;
(E) $n = 4$, $a = 3$, $\lambda = 3$, $g = 0$, $d = 1$, $\Delta = 5$.

Proposition 3.2. *If* $r = 1$, *then one of the following cases holds:*

(I) $n = 3$, $a = 1$, $\mathfrak{B}$ *is a conic;*
(II) $n = 4$, $a = 0$, $\mathfrak{B}$ *is an elliptic curve of degree* 5*;*
(III) $n = 4$, $a = 1$, $\mathfrak{B}$ *is the rational normal quartic curve;*
(IV) $n = 4$, $a = 3$, $\mathfrak{B}$ *is the twisted cubic curve.*

Proof. From Lemma 3.1 it remains only to exclude case (D). In this case $\mathfrak{B}$ is a complete intersection of two quadrics in $\mathbb{P}^3$ and also it is an *OADP*-curve. This is absurd because the only *OADP*-curve is the twisted cubic curve. □

4. Case of dimension 2

Proposition 4.1 follows from [37, Propositions 1.3 and 3.4] and [11, Theorem 4.10].

Proposition 4.1. *If* $r = 2$, *then either* $n = 6$, $d \geq 2$, $\langle \mathfrak{B} \rangle = \mathbb{P}^6$, *or one of the following cases holds:*

(V) $n = 4$, $d = 1$, $\delta = 2$, $\mathfrak{B} = \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3 \subset \mathbb{P}^4$*;*
(VI) $n = 5$, $d = 1$, $\delta = 1$, $\mathfrak{B}$ *is a hyperplane section of* $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$*;*
(VII) $n = 5$, $d = 2$, $\delta = 1$, $\mathfrak{B} = \nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$ *is the Veronese surface;*
(VIII) $n = 6$, $d = 1$, $\delta = 0$, $\mathfrak{B} \subset \mathbb{P}^5$ *is an OADP-surface, i.e.* $\mathfrak{B}$ *is as in one of the following cases:*
  (VIII$_1$) $\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1) \oplus \mathscr{O}(3))$ *or* $\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(2) \oplus \mathscr{O}(2))$*;*
  (VIII$_2$) del Pezzo surface of degree 5 *(hence the blow-up of* $\mathbb{P}^2$ *at* 4 *points* $p_1, \ldots, p_4$ *and* $|H_{\mathfrak{B}}| = |3H_{\mathbb{P}^2} - p_1 - \cdots - p_4|$*).*

Lemma 4.2. *If* $r = 2$, $n = 6$ *and* $\langle \mathfrak{B} \rangle = \mathbb{P}^6$, *then one of the following cases holds:*

(A) $a = 0$, $\lambda = 7$, $g = 1$, $\chi(\mathscr{O}_{\mathfrak{B}}) = 0$;
(B) $0 \leq a \leq 3$, $\lambda = 8 - a$, $g = 3 - a$, $\chi(\mathscr{O}_{\mathfrak{B}}) = 1$.

Proof. By Proposition 2.1 it follows that $g = 2\lambda + a - 13$ and $\chi(\mathscr{O}_{\mathfrak{B}}) = \lambda + a - 7$. By [40, Lemma 6.1] and using that $g \geq 0$ (proceeding as in [40, Proposition 6.2]), we obtain $(13-a)/2 \leq \lambda \leq 8 - a$. □

Lemma 4.3. *If $r=2$, $n=6$ and $\langle\mathfrak{B}\rangle=\mathbb{P}^6$, then one of the following cases holds:*

- $a=0$, $d=4$, $\Delta=1$;
- $a=1$, $d=3$, $\Delta=2$;
- $a=2$, $d=2$, $\Delta=4$;
- $a=3$, $d=2$, $\Delta=5$.

Proof. We have $s_1(\mathscr{T}_{\mathfrak{B}})\cdot H_{\mathfrak{B}}=-c_1$ and $s_2(\mathscr{T}_{\mathfrak{B}})=c_1^2-c_2=12\chi(\mathscr{O}_{\mathfrak{B}})-2c_2$. So, by Proposition 2.4, we obtain

$$2(2d-1)=\lambda^2-10\lambda-12\chi(\mathscr{O}_{\mathfrak{B}})+2c_2+5c_1. \tag{4.1}$$

Now, by Propositions 2.1 and 2.2, we obtain

$$d\Delta=2a+4,\quad \Delta=(g^2+(-2a-4)g-16d+a^2-4a+75)/8, \tag{4.2}$$

and then we conclude by Lemma 4.2. □

Proposition 4.4. *If $r=2$, $n=6$ and $\langle\mathfrak{B}\rangle=\mathbb{P}^6$ then one of the following cases holds:*

(IX) $a=0$, $\lambda=7$, $g=1$, $\mathfrak{B}$ *is an elliptic scroll* $\mathbb{P}_C(\mathscr{E})$ *with* $e(\mathscr{E})=-1$;

(X) $a=0$, $\lambda=8$, $g=3$, $\mathfrak{B}$ *is the blow-up of* $\mathbb{P}^2$ *at* 8 *points* $p_1\ldots,p_8$, $|H_{\mathfrak{B}}|=|4H_{\mathbb{P}^2}-p_1-\cdots-p_8|$;

(XI) $a=1$, $\lambda=7$, $g=2$, $\mathfrak{B}$ *is the blow-up of* $\mathbb{P}^2$ *at* 6 *points* $p_0\ldots,p_5$, $|H_{\mathfrak{B}}|=|4H_{\mathbb{P}^2}-2p_0-p_1-\cdots-p_5|$;

(XII) $a=2$, $\lambda=6$, $g=1$, $\mathfrak{B}$ *is the blow-up of* $\mathbb{P}^2$ *at* 3 *points* p_1,p_2,p_3, $|H_{\mathfrak{B}}|=|3H_{\mathbb{P}^2}-p_1-p_2-p_3|$;

(XIII) $a=3$, $\lambda=5$, $g=0$, $\mathfrak{B}$ *is a rational normal scroll.*

Proof. For $a=0$, $a=1$ and $a\in\{2,3\}$ the statement follows, respectively, from [12], [40, Proposition 6.2] and [25]. □

5. Case of dimension 3

Proposition 5.1 follows from: [37, Proposition 1.3 and 3.4], [18], [30], [19, page 62] and [11].

Proposition 5.1. *If $r=3$, then either $n=8$, $d\geq 2$, $\langle\mathfrak{B}\rangle=\mathbb{P}^8$, or one of the following cases holds:*

(XIV) $n=5$, $d=1$, $\delta=3$, $\mathfrak{B}=Q^3\subset\mathbb{P}^4\subset\mathbb{P}^5$ *is a quadric;*

(XV) $n=6$, $d=1$, $\delta=2$, $\mathfrak{B}=\mathbb{P}^1\times\mathbb{P}^2\subset\mathbb{P}^5\subset\mathbb{P}^6$;

(XVI) $n=7$, $d=1$, $\delta=1$, $\mathfrak{B}\subset\mathbb{P}^6$ *is as in one of the following cases:*

- *(XVI_1)* $\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1)\oplus\mathscr{O}(1)\oplus\mathscr{O}(2))$;
- *(XVI_2) linear section of* $\mathbb{G}(1,4)\subset\mathbb{P}^9$;

(XVII) $n=7$, $d=2$, $\delta=1$, $\mathfrak{B}$ *is a hyperplane section of* $\mathbb{P}^2\times\mathbb{P}^2\subset\mathbb{P}^8$;

(XVIII) $n=8$, $d=1$, $\delta=0$, $\mathfrak{B}\subset\mathbb{P}^7$ *is an OADP-variety, i.e.* $\mathfrak{B}$ *is as in one of the following cases:*

- *($XVIII_1$)* $\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1)\oplus\mathscr{O}(1)\oplus\mathscr{O}(3))$ *or* $\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1)\oplus\mathscr{O}(2)\oplus\mathscr{O}(2))$;
- *($XVIII_2$) Edge variety of degree* 6 *(i.e.* $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$*) or Edge variety of degree* 7;
- *($XVIII_3$)* $\mathbb{P}_{\mathbb{P}^2}(\mathscr{E})$, *where* $\mathscr{E}$ *is a vector bundle with* $c_1(\mathscr{E})=4$ *and* $c_2(\mathscr{E})=8$, *given as an extension by the following exact sequence* $0\to\mathscr{O}_{\mathbb{P}^2}\to\mathscr{E}\to\mathscr{I}_{\{p_1,\ldots,p_8\},\mathbb{P}^2}(4)\to 0$.

In the following we denote by $\Lambda\subsetneq C\subsetneq S\subsetneq\mathfrak{B}$ a sequence of general linear sections of $\mathfrak{B}$.

Lemma 5.2. *If $r = 3$, $n = 8$ and $\langle \mathfrak{B} \rangle = \mathbb{P}^8$, then one of the following cases holds:*
(A) $a = 0$, $\lambda = 13$, $g = 8$, $K_S \cdot H_S = 1$, $K_S^2 = -1$;
(B) $a = 1$, $\lambda = 12$, $g = 7$, $K_S \cdot H_S = 0$, $K_S^2 = 0$;
(C) $0 \leq a \leq 6$, $\lambda = 12 - a$, $g = 6 - a$, $K_S \cdot H_S = -2 - a$.

Proof. Firstly we note that, from the exact sequence $0 \to \mathscr{T}_S \to \mathscr{T}_{\mathfrak{B}}|_S \to \mathscr{O}_S(1) \to 0$, we deduce $c_2 = c_2(S) + c_1(S) = 12\chi(\mathscr{O}_S) - K_S^2 - K_S \cdot H_S$ and hence

$$K_S^2 = 14\lambda + 12\chi(\mathscr{O}_S) - 12g + d\Delta - 116 = -22\lambda + 12g + d\Delta - 12a + 184. \tag{5.1}$$

Secondly we note that (see [40, Lemma 6.1]), putting $h_\Lambda(2) := h^0(\mathbb{P}^5, \mathscr{O}(2)) - h^0(\mathbb{P}^5, \mathscr{I}_\Lambda(2))$, we have

$$\min\{\lambda, 11\} \leq h_\Lambda(2) \leq 21 - h^0(\mathbb{P}^8, \mathscr{I}_{\mathfrak{B}}(2)) = 12 - a. \tag{5.2}$$

Now we establish the following:

Claim 5.2.1. If $K_S \cdot H_S \leq 0$ and $K_S \nsim 0$, then $\lambda = 12 - a$ and $g = 6 - a$.

Proof of the Claim. Similarly to [40, Case 6.1], we obtain that $P_{\mathfrak{B}}(-1) = 0$ and $P_{\mathfrak{B}}(0) = 1 - q$, where $q := h^1(S, \mathscr{O}_S) = h^1(\mathfrak{B}, \mathscr{O}_{\mathfrak{B}})$; in particular $g = -5q - a + 6$ and $\lambda = -3q - a + 12$. Since $g \geq 0$ we have $5q \leq 6 - a$ and the possibilities are: if $a \leq 1$ then $q \leq 1$; if $a \geq 2$ then $q = 0$. If $(a, q) = (0, 1)$ then $(g, \lambda) = (1, 9)$ and the case is excluded by [19, Theorem 12.3][2]; if $(a, q) = (1, 1)$ then $(g, \lambda) = (0, 8)$ and the case is excluded by [19, Theorem 12.1]. Thus we have $q = 0$ and hence $g = 6 - a$ and $\lambda = 12 - a$; in particular we have $a \leq 6$. □

Now we discuss the cases according to the value of a.

Case 5.2.1 ($a = 0$). It is clear that φ must be of type $(2, 5)$ and hence $K_S^2 = -22\lambda + 12g + 189$. By Claim 5.2.1, if $K_S \cdot H_S = 2g - 2 - \lambda < 0$, we fall into case (C). So we suppose that $K_S \cdot H_S \geq 0$, namely that $g \geq \lambda/2 + 1$. From Castelnuovo's bound it follows that $\lambda \geq 12$ and if $\lambda = 12$ then $K_S \cdot H_S = 0$, $g = 7$ and hence $K_S^2 = 9$. Since this is impossible by Claim 5.2.1, we conclude that $\lambda \geq 13$. Now by (5.2) it follows that $11 \leq h_\Lambda(2) \leq 12$, but if $h_\Lambda(2) = 11$ from Castelnuovo Lemma [10, Lemma 1.10] we obtain a contradiction. Thus we have $h_\Lambda(2) = 12$ and $h^0(\mathbb{P}^5, \mathscr{I}_\Lambda(2)) = h^0(\mathbb{P}^8, \mathscr{I}_{\mathfrak{B}}(2)) = 9$. So from [10, Theorem 3.1] we deduce that $\lambda \leq 14$ and furthermore, by the refinement of Castelnuovo's bound contained in [10, Theorem 2.5], we obtain $g \leq 2\lambda - 18$. In summary we have the following possibilities:

(i) $\lambda = 13$, $g = 8$, $K_S \cdot H_S = 1$, $\chi(\mathscr{O}_S) = 2$, $K_S^2 = -1$;
(ii) $\lambda = 14$, $g = 8$, $K_S \cdot H_S = 0$, $\chi(\mathscr{O}_S) = -1$, $K_S^2 = -23$;
(iii) $\lambda = 14$, $g = 9$, $K_S \cdot H_S = 2$, $\chi(\mathscr{O}_S) = 1$, $K_S^2 = -11$;
(iv) $\lambda = 14$, $g = 10$, $K_S \cdot H_S = 4$, $\chi(\mathscr{O}_S) = 3$, $K_S^2 = 1$.

Case (i) coincides with case (A). Case (ii) is excluded by Claim 5.2.1. In the circumstances of case (iii), we have $h^1(S, \mathscr{O}_S) = h^2(S, \mathscr{O}_S) = h^0(S, K_S)$. If $h^1(S, \mathscr{O}_S) > 0$, since $(K_{\mathfrak{B}} + 4H_{\mathfrak{B}}) \cdot K_S = K_S^2 + 3K_S \cdot H_S = -5 < 0$, we see that $K_{\mathfrak{B}} + 4H_{\mathfrak{B}}$ is not nef and then we obtain a contradiction by [28]. If $h^1(S, \mathscr{O}_S) = 0$, then we also have $h^1(\mathfrak{B}, \mathscr{O}_{\mathfrak{B}}) = h^2(\mathfrak{B}, \mathscr{O}_{\mathfrak{B}}) = 0$ and hence $\chi(\mathscr{O}_{\mathfrak{B}}) = 1 - h^3(\mathfrak{B}, \mathscr{O}_{\mathfrak{B}}) \leq 1$, against the fact that $\chi(\mathscr{O}_{\mathfrak{B}}) = 2\lambda - g - 17 = 2$. Thus case (iii) does not occur. Finally, in the circumstances of case (iv), note that $h^0(S, K_S) = 2 + h^1(S, \mathscr{O}_S) \geq 2$ and

[2] Note that $\mathfrak{B}$ cannot be a scroll over a curve (this follows from (5.8) and (5.9) below and also it follows from [33, Proposition 3.2(i)]).

we write $|K_S| = |M| + F$, where $|M|$ is the mobile part of the linear system $|K_S|$ and F is the fixed part. If $M_1 = M$ is a general member of $|M|$, there exists $M_2 \in |M|$ having no common irreducible components with M_1 and so $M^2 = M_1 \cdot M_2 = \sum_p (M_1 \cdot M_2)_p \geq 0$; furthermore, by using Bertini Theorem, we see that $\mathrm{sing}(M_1)$ consists of points p such that the intersection multiplicity $(M_1 \cdot M_2)_p$ of M_1 and M_2 in p is at least 2. By definition, we also have $M \cdot F \geq 0$ and so we deduce $2p_a(M) - 2 = M \cdot (M + K_S) = 2M^2 + M \cdot F \geq 0$, from which $p_a(M) \geq 1$ and $p_a(M) = 2$ if $F = 0$. On the other hand, we have $M \cdot H_S \leq K_S \cdot H_S = 4$ and, since S is cut out by quadrics, M does not contain planar curves of degree ≥ 3. If $M \cdot H_S = 4$, then $F = 0$, $M^2 = 1$ and M is a (possibly disconnected) smooth curve; since $p_a(M) = 2$, M is actually disconnected and so it is a disjoint union of twisted cubics, conics and lines. But then we obtain the contradiction that $p_a(M) = 1 - \#\{\text{connected components of } M\} < 0$. If $M \cdot H_S \leq 3$, then M must be either a twisted cubic or a union of conics and lines. In all these cases we again obtain the contradiction that $p_a(M) = 1 - \#\{\text{connected components of } M\} \leq 0$. Thus case (iv) does not occur.

Case 5.2.2 ($a = 1$). By [40, Proposition 6.4] we fall into case (B) or (C).

Case 5.2.3 ($a \geq 2$). By (5.2) it follows that $\lambda \leq 10$ and by Castelnuovo's bound it follows that $K_S \cdot H_S \leq -4 < 0$. Thus, by Claim 5.2.1 we fall into case (C).

□

Now we apply the double point formula (Proposition 2.4) in order to obtain additional numerical restrictions under the hypothesis of Lemma 5.2.

Lemma 5.3. *If $r = 3$, $n = 8$ and $\langle \mathfrak{B} \rangle = \mathbb{P}^8$, then*
$$K_{\mathfrak{B}}^3 = \lambda^2 + 23\lambda - 24g - (7d+1)\Delta - 4d + 36a - 226.$$

Proof. We have (see [23, App. A, Exercise 6.7]):
$$\begin{aligned}
s_1(\mathscr{T}_{\mathfrak{B}}) \cdot H_{\mathfrak{B}}^2 &= -c_1(\mathfrak{B}) \cdot H_{\mathfrak{B}}^2 = K_{\mathfrak{B}} \cdot H_{\mathfrak{B}}^2, \\
s_2(\mathscr{T}_{\mathfrak{B}}) \cdot H_{\mathfrak{B}} &= c_1(\mathfrak{B})^2 \cdot H_{\mathfrak{B}} - c_2(\mathfrak{B}) \cdot H_{\mathfrak{B}} = K_{\mathfrak{B}}^2 \cdot H_{\mathfrak{B}} - c_2(\mathfrak{B}) \cdot H_{\mathfrak{B}} \\
&= 3K_{\mathfrak{B}} \cdot H_{\mathfrak{B}}^2 - 2H_{\mathfrak{B}}^3 - 2c_2(\mathfrak{B}) \cdot H_{\mathfrak{B}} + 12\left(\chi(\mathscr{O}_{\mathfrak{B}}(H_{\mathfrak{B}})) - \chi(\mathscr{O}_{\mathfrak{B}})\right), \\
s_3(\mathscr{T}_{\mathfrak{B}}) &= -c_1(\mathfrak{B})^3 + 2c_1(\mathfrak{B}) \cdot c_2(\mathfrak{B}) - c_3(\mathfrak{B}) = K_{\mathfrak{B}}^3 + 48\chi(\mathscr{O}_{\mathfrak{B}}) - c_3(\mathfrak{B}).
\end{aligned}$$
Hence, applying the double point formula and using the relations $\chi(\mathscr{O}_{\mathfrak{B}}) = 2\lambda - g + a - 17$, $\chi(\mathscr{O}_{\mathfrak{B}}(H_{\mathfrak{B}})) = 9$, we obtain:
$$\begin{aligned}
4d - 2 &= 2\deg(\mathrm{Sec}(\mathfrak{B})) \\
&= \deg(\mathfrak{B})^2 - s_3(\mathscr{T}_{\mathfrak{B}}) - 7\,s_2(\mathscr{T}_{\mathfrak{B}}) \cdot H_{\mathfrak{B}} - 21\,s_1(\mathscr{T}_{\mathfrak{B}}) \cdot H_{\mathfrak{B}}^2 - 35\,H_{\mathfrak{B}}^3 \\
&= \deg(\mathfrak{B})^2 - 21\deg(\mathfrak{B}) - 42\,K_{\mathfrak{B}} \cdot H_{\mathfrak{B}}^2 + 14\,c_2(\mathfrak{B}) \cdot H_{\mathfrak{B}} - K_{\mathfrak{B}}^3 \\
&\quad + c_3(\mathfrak{B}) - 84\,\chi(\mathscr{O}_{\mathfrak{B}}(H_{\mathfrak{B}})) + 36\,\chi(\mathscr{O}_{\mathfrak{B}}) \\
&= -K_{\mathfrak{B}}^3 + \lambda^2 + 23\lambda - 24g - (7d+1)\Delta + 36a - 228.
\end{aligned}$$

□

Lemma 5.4. *If $r = 3$, $n = 8$, $\langle \mathfrak{B} \rangle = \mathbb{P}^8$ and $\mathfrak{B}$ is a quadric fibration over a curve, then one of the following cases holds:*

- $a = 3$, $\lambda = 9$, $g = 3$, $d = 3$, $\Delta = 5$;
- $a = 4$, $\lambda = 8$, $g = 2$, $d = 2$, $\Delta = 10$.

Proof. Denote by $\beta : (\mathfrak{B}, H_{\mathfrak{B}}) \to (Y, H_Y)$ the projection over the curve Y such that $\beta^*(H_Y) = K_{\mathfrak{B}} + 2H_{\mathfrak{B}}$. We have

$$\begin{array}{rcl} 0 & = & \beta^*(H_Y)^2 \cdot H_{\mathfrak{B}} = K_{\mathfrak{B}}^2 \cdot H_{\mathfrak{B}} + 4K_{\mathfrak{B}} \cdot H_{\mathfrak{B}}^2 + 4H_{\mathfrak{B}}^3, \\ 0 & = & \beta^*(H_Y)^3 = K_{\mathfrak{B}}^3 + 6K_{\mathfrak{B}}^2 \cdot H_{\mathfrak{B}} + 12K_{\mathfrak{B}} \cdot H_{\mathfrak{B}}^2 + 8H_{\mathfrak{B}}^3, \\ \chi(\mathscr{O}_{\mathfrak{B}}(H_{\mathfrak{B}})) & = & \frac{1}{12}K_{\mathfrak{B}}^2 \cdot H_{\mathfrak{B}} - \frac{1}{4}K_{\mathfrak{B}} \cdot H_{\mathfrak{B}}^2 + \frac{1}{6}H_{\mathfrak{B}}^3 + \frac{1}{12}c_2(\mathfrak{B}) \cdot H_{\mathfrak{B}} + \chi(\mathscr{O}_{\mathfrak{B}}), \end{array}$$

from which it follows that

$$K_{\mathfrak{B}}^3 = -8\lambda + 24g - 24, \tag{5.3}$$

$$c_2(\mathfrak{B}) \cdot H_{\mathfrak{B}} = -36\lambda + 26g - 12a + 298. \tag{5.4}$$

Hence, by Lemma 5.3 and Proposition 2.2, we obtain

$$d\Delta = 23\lambda - 16g + 12a - 180, \tag{5.5}$$

$$\Delta + 4d = \lambda^2 - 130\lambda + 64g - 48a + 1058. \tag{5.6}$$

Now the conclusion follows from Lemma 5.2, by observing that the case $a = 6$ cannot occur. In fact, if $a = 6$, by [25] it follows that $\mathfrak{B}$ is a rational normal scroll and by a direct calculation (or by Lemma 5.6) we see that $d = 2$ and $\Delta = 14$. □

Lemma 5.5. *If $r = 3$, $n = 8$, $\langle \mathfrak{B} \rangle = \mathbb{P}^8$ and $\mathfrak{B}$ is a scroll over a smooth surface Y, then we have:*

$$\begin{array}{rcl} c_2(Y) & = & \left((7d-1)\lambda^2 + (177 - 679d)\lambda + (292d - 92)g - 28d^2 \right. \\ & & \left. + (5554 - 252a)d + 36a - 1474\right) / (2d+2), \\ \Delta & = & \left(\lambda^2 - 107\lambda + 48g - 4d - 36a + 878\right) / (d+1). \end{array}$$

Proof. Similarly to Lemma 5.4, denote by $\beta : (\mathfrak{B}, H_{\mathfrak{B}}) \to (Y, H_Y)$ the projection over the surface Y such that $\beta^*(H_Y) = K_{\mathfrak{B}} + 2H_{\mathfrak{B}}$. Since $\beta^*(H_Y)^3 = 0$ we obtain

$$\begin{array}{rcl} K_{\mathfrak{B}}^3 & = & -8H_{\mathfrak{B}}^3 - 12K_{\mathfrak{B}} \cdot H_{\mathfrak{B}}^2 - 6K_{\mathfrak{B}}^2 \cdot H_{\mathfrak{B}} \\ & = & -30K_{\mathfrak{B}} \cdot H_{\mathfrak{B}}^2 + 4H_{\mathfrak{B}}^3 + 6c_2(\mathfrak{B}) \cdot H_{\mathfrak{B}} - 72\chi(\mathscr{O}_{\mathfrak{B}}(H_{\mathfrak{B}})) + 72\chi(\mathscr{O}_{\mathfrak{B}}) \\ & = & 130\lambda - 72g - 6d\Delta + 72a - 1104. \end{array}$$

Now we conclude comparing the last formula with Lemma 5.3 and using the relation

$$70\lambda - 44g + (7d-1)\Delta - 596 = c_3(\mathfrak{B}) = c_1(\mathbb{P}^1)c_2(Y) = 2c_2(Y). \tag{5.7}$$

□

Lemma 5.6. *If $r = 3$, $n = 8$, $\langle \mathfrak{B} \rangle = \mathbb{P}^8$ and $\mathfrak{B}$ is a scroll over a smooth curve, then we have: $a = 6$, $\lambda = 6$, $g = 0$, $d = 2$, $\Delta = 14$.*

Proof. We have a projection $\beta : (\mathfrak{B}, H_{\mathfrak{B}}) \to (Y, H_Y)$ over a curve Y such that $\beta^*(H_Y) = K_{\mathfrak{B}} + 3H_{\mathfrak{B}}$. By expanding the expressions $\beta^*(H_Y)^2 \cdot H_{\mathfrak{B}} = 0$ and $\beta^*(H_Y)^3 = 0$ we obtain $K_{\mathfrak{B}}^2 \cdot H_{\mathfrak{B}} = 3\lambda - 12g + 12$ and $K_{\mathfrak{B}}^3 = 54(g-1)$, and hence by Lemma 5.3 we get

$$\lambda^2 + 23\lambda - 78g - (7d+1)\Delta - 4d + 36a - 172 = 0. \tag{5.8}$$

Also, by expanding the expression $\chi(\mathscr{O}_{\mathfrak{B}}(H_{\mathfrak{B}})) = 9$ we obtain $c_2 = -35\lambda + 30g - 12a + 294$ and hence by Proposition 2.2 we get

$$22\lambda - 20g - d\Delta + 12a - 176 = 0. \tag{5.9}$$

Now the conclusion follows from Lemma 5.2. □

Finally we conclude our discussion about classification with the following:

Proposition 5.7. *If $r=3$, $n=8$ and $\langle\mathfrak{B}\rangle=\mathbb{P}^8$, then one of the following cases holds:*

(XIX) $a=0$, $\lambda=12$, $g=6$, *$\mathfrak{B}$ is a scroll $\mathbb{P}_Y(\mathscr{E})$ over a birationally ruled surface Y with $K_Y^2=5$, $c_2(\mathscr{E})=8$ and $c_1^2(\mathscr{E})=20$;*

(XX) $a=0$, $\lambda=13$, $g=8$, *$\mathfrak{B}$ is obtained as the blow-up of a Fano variety X at a point $p\in X$, $|H_{\mathfrak{B}}|=|H_X-p|$;*

(XXI) $a=1$, $\lambda=11$, $g=5$, *$\mathfrak{B}$ is the blow-up of Q^3 at 5 points $p_1,\ldots,p_5$, $|H_{\mathfrak{B}}|=|2H_{Q^3}-p_1-\cdots-p_5|$;*

(XXII) $a=1$, $\lambda=11$, $g=5$, *$\mathfrak{B}$ is a scroll over $\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}\oplus\mathscr{O}(-1))$;*

(XXIII) $a=1$, $\lambda=12$, $g=7$, *$\mathfrak{B}$ is a linear section of $S^{10}\subset\mathbb{P}^{15}$;*

(XXIV) $a=2$, $\lambda=10$, $g=4$, *$\mathfrak{B}$ is a scroll over Q^2;*

(XXV) $a=3$, $\lambda=9$, $g=3$, *$\mathfrak{B}$ is a scroll over $\mathbb{P}^2$ or a quadric fibration over $\mathbb{P}^1$;*

(XXVI) $a=4$, $\lambda=8$, $g=2$, *$\mathfrak{B}$ is a hyperplane section of $\mathbb{P}^1\times Q^3$;*

(XXVII) $a=6$, $\lambda=6$, $g=0$, *$\mathfrak{B}$ is a rational normal scroll.*

Proof. For $a=6$ the statement follows from [25]. The case with $a=5$ is excluded by [25] and Example 6.19. For $a=4$ the statement follows from [29]. For $a\in\{2,3\}$, by [16], [17] and [27] it follows that the abstract structure of $\mathfrak{B}$ is as asserted, or $a=2$ and $\mathfrak{B}$ is a quadric fibration over $\mathbb{P}^1$; the last case is excluded by Lemma 5.4. For $a=1$ the statement is just [40, Proposition 6.6]. Now we treat the cases with $a=0$.

Case 5.7.1 ($a=0,\lambda=12$). Since $\deg(\mathfrak{B})\leq 2\mathrm{codim}_{\mathbb{P}^8}(\mathfrak{B})+2$, it follows that $(\mathfrak{B},H_{\mathfrak{B}})$ must be as in one of the cases (a),...,(h) of [26, Theorem 1]. Cases (a), (d), (e), (g), (h) are of course impossible and case (c) is excluded by Lemma 5.4. If $\mathfrak{B}$ is as in case (b), by Lemma 5.6 we obtain that $\mathfrak{B}$ is a scroll over a birationally ruled surface. Now suppose that $(\mathfrak{B},H_{\mathfrak{B}})$ is as in case (f). Thus there is a reduction (X,H_X) as in one of the cases:

(f1) $X=\mathbb{P}^3$, $H_X\in|\mathscr{O}(3)|$;
(f2) $X=Q^3$, $H_X\in|\mathscr{O}(2)|$;
(f3) X is a $\mathbb{P}^2$-bundle over a smooth curve such that $\mathscr{O}_X(H_X)$ induces $\mathscr{O}(2)$ on each fiber.

By definition of reduction we have $X\subset\mathbb{P}^N$, where $N=8+s$, $\deg(X)=\lambda+s=12+s$ and s is the number of points blown up on X to get $\mathfrak{B}$. Case (f1) and (f2) are impossible because they force λ to be respectively 16 and 11. In case (f3), we have a projection $\beta:(X,H_X)\to(Y,H_Y)$ over a curve Y such that $\beta^*(H_Y)=2K_X+3H_X$. Hence we get

$$K_XH_X^2=(2K_X+3H_X)^2\cdot H_X/12-K_X^2\cdot H_X/3-3H_X^3/4=-K_X^2\cdot H_X/3-3H_X^3/4,$$

from which we deduce that

$$\begin{aligned}0 &= (2K_X+3H_X)^3=8K_X^3+36K_X^2\cdot H_X+54K_X\cdot H_X^2+27H_X^3\\ &= 8K_X^3+18K_X^2\cdot H_X-27H_X^3/2\\ &= 8(K_{\mathfrak{B}}^3-8s)+18K_X^2\cdot H_X-27(\deg(\mathfrak{B})+s)/2\\ &= 18K_X^2\cdot H_X-155s/2-210.\end{aligned}$$

Since $s\leq 12$ (see [7, Lemma 8.1]), we conclude that case (f) does not occur. Thus, $\mathfrak{B}=\mathbb{P}_Y(\mathscr{E})$ is a scroll over a surface Y; moreover, by Lemma 5.5 and [5, Theorem 11.1.2], we obtain $K_Y^2=5$, $c_2(\mathscr{E})=K_Y^2-K_S^2=8$ and $c_1^2(\mathscr{E})=\lambda+c_2(\mathscr{E})=20$.

Case 5.7.2 ($a=0,\lambda=13$). The proof is located in [33, page 16], but we sketch it for the reader's convenience. By Lemma 5.2 we know that $\chi(\mathscr{O}_S)=2$ and K_S is an exceptional curve of the first kind. Thus, if we blow-down the divisor K_S, we obtain a $K3$-surface. By using adjunction theory (see for instance [5] or Ionescu's papers cited in the references) and by Lemmas 5.4, 5.5 and 5.6 it follows that the adjunction map $\phi_{|K_{\mathfrak{B}}+2H_{\mathfrak{B}}|}$ is a generically finite morphism; moreover, since $(K_{\mathfrak{B}}+2H_{\mathfrak{B}})\cdot K_S=0$, we see that $\phi_{|K_{\mathfrak{B}}+2H_{\mathfrak{B}}|}$ is not a finite morphism. So, we deduce that there is a $(\mathbb{P}^2,\mathscr{O}_{\mathbb{P}^2}(-1))$ inside $\mathfrak{B}$ and, after the blow-down of this divisor, we get a smooth Fano 3-fold $X\subset\mathbb{P}^9$ of sectional genus 8 and degree 14.

□

6. EXAMPLES

The calculations in the following examples can be verified with the aid of the computer algebra system [20].

Example 6.1 ($r=1,2,3;n=3,4,5;a=1;d=1$)**.** See also [40, §2]. If $Q\subset\mathbb{P}^{n-1}\subset\mathbb{P}^n$ is a smooth quadric, then the linear system $|\mathscr{I}_{Q,\mathbb{P}^n}(2)|$ defines a birational transformation $\psi:\mathbb{P}^n\dashrightarrow\mathbf{S}\subset\mathbb{P}^{n+1}$ of type $(2,1)$ whose image is a smooth quadric.

Example 6.2 ($r=1;n=4;a=0;d=3$)**.** See also [12]. If $X\subset\mathbb{P}^4$ is a nondegenerate curve of genus 1 and degree 5, then X is the scheme-theoretic intersection of the quadrics (of rank 3) containing X and $|\mathscr{I}_{X,\mathbb{P}^4}(2)|$ defines a Cremona transformation $\mathbb{P}^4\dashrightarrow\mathbb{P}^4$ of type $(2,3)$.

Example 6.3 ($r=1,2,3;n=4,5,7;a=1,0,1;d=2$)**.** See also [14] and [40, Example 4.1]. If $X\subset\mathbb{P}^n$ is a Severi variety, then $|\mathscr{I}_{X,\mathbb{P}^n}(2)|$ defines a birational transformation $\psi:\mathbb{P}^n\dashrightarrow\mathbb{P}^n$ of type $(2,2)$ whose base locus is X. The restriction of ψ to a general hyperplane is a birational transformation $\mathbb{P}^{n-1}\dashrightarrow\mathbf{S}\subset\mathbb{P}^n$ of type $(2,2)$ and $\mathbf{S}$ is a smooth quadric.

Example 6.4 ($r=1;n=4;a=2;d=1$ - not satisfying 1.3)**.** We have a special birational transformation $\psi:\mathbb{P}^4\dashrightarrow\mathbf{S}\subset\mathbb{P}^6$ of type $(2,1)$ with base locus X, image $\mathbf{S}$ and base locus of the inverse Y, as follows:

$$\begin{aligned}
X &= V(x_0x_1-x_2^2-x_3^2,-x_0^2-x_1^2+x_2x_3,x_4),\\
\mathbf{S} &= V(y_2y_3-y_4^2-y_5^2-y_0y_6,y_2^2+y_3^2-y_4y_5+y_1y_6),\\
P_{\mathbf{S}}(t) &= (4t^4+24t^3+56t^2+60t+24)/4!,\\
\mathrm{sing}(\mathbf{S}) &= V(y_6,y_5^2,y_4y_5,y_3y_5,y_2y_5,y_4^2,y_3y_4,y_2y_4,2y_1y_4+y_0y_5,\\
&\qquad y_0y_4+2y_1y_5,y_3^2,y_2y_3,y_2^2,y_1y_2+2y_0y_3,2y_0y_2+y_1y_3),\\
P_{\mathrm{sing}(\mathbf{S})}(t) &= t+5,\\
(\mathrm{sing}(\mathbf{S}))_{\mathrm{red}} &= V(y_6,y_5,y_4,y_3,y_2),\\
Y=(Y)_{\mathrm{red}} &= (\mathrm{sing}(\mathbf{S}))_{\mathrm{red}}=V(y_6,y_5,y_4,y_3,y_2).
\end{aligned}$$

See also [40, Example 4.6] for another example in which 1.3 is not satisfied.

Example 6.5 ($r=1,2,3;n=4,5,6;a=3;d=1$)**.** See also [38] and [39]. If $X=\mathbb{P}^1\times\mathbb{P}^2\subset\mathbb{P}^5\subset\mathbb{P}^6$, then $|\mathscr{I}_{X,\mathbb{P}^6}(2)|$ defines a birational transformation $\psi:\mathbb{P}^6\dashrightarrow\mathbf{S}\subset\mathbb{P}^9$ of type $(2,1)$ whose base locus is X and whose image is $\mathbf{S}=\mathbb{G}(1,4)$. Restricting ψ to a general $\mathbb{P}^5\subset\mathbb{P}^6$ (resp. $\mathbb{P}^4\subset\mathbb{P}^6$) we obtain a birational transformation $\mathbb{P}^5\dashrightarrow\mathbf{S}\subset\mathbb{P}^8$ (resp. $\mathbb{P}^4\dashrightarrow\mathbf{S}\subset\mathbb{P}^7$) whose image is a smooth linear section of $\mathbb{G}(1,4)\subset\mathbb{P}^9$.

Example 6.6 ($r=2;n=6;a=0;d=4$)**.** See also [12] and [24]. Let $Z=\{p_1,\ldots,p_8\}\subset\mathbb{P}^2$ be such that no 4 of the p_i are collinear and no 7 of the p_i lie on a conic and consider the blow-up $X=\mathrm{Bl}_Z(\mathbb{P}^2)$ embedded in $\mathbb{P}^6$ by $|4H_{\mathbb{P}^2}-p_1-\cdots-p_8|$. Then the homogeneous ideal of X is generated by quadrics and $|\mathscr{I}_{X,\mathbb{P}^6}(2)|$ defines a Cremona transformation $\mathbb{P}^6 \dashrightarrow \mathbb{P}^6$ of type $(2,4)$. The same happens when $X\subset\mathbb{P}^6$ is a septic elliptic scroll with $e=-1$.

Example 6.7 ($r=2;n=6;a=1;d=3$)**.** See also [40, Examples 4.2 and 4.3]. If $X\subset\mathbb{P}^6$ is a general hyperplane section of an Edge variety of dimension 3 and degree 7 in $\mathbb{P}^7$, then $|\mathscr{I}_{X,\mathbb{P}^6}(2)|$ defines a birational transformation $\psi:\mathbb{P}^6\dashrightarrow\mathbf{S}\subset\mathbb{P}^7$ of type $(2,3)$ whose base locus is X and whose image is a rank 6 quadric.

Example 6.8 ($r=2;n=6;a=2;d=2$)**.** If $X\subset\mathbb{P}^6$ is the blow-up of $\mathbb{P}^2$ at 3 general points p_1,p_2,p_3 with $|H_X|=|3H_{\mathbb{P}^2}-p_1-p_2-p_3|$, then $\mathrm{Sec}(X)$ is a cubic hypersurface. By Fact 1.5 and 1.6 we deduce that $|\mathscr{I}_{X,\mathbb{P}^6}(2)|$ defines a birational transformation $\psi:\mathbb{P}^6\dashrightarrow\mathbf{S}\subset\mathbb{P}^8$ and its type is $(2,2)$. The image $\mathbf{S}$ is a complete intersection of two quadrics, $\dim(\mathrm{sing}(\mathbf{S}))=1$ and the base locus of the inverse is $\mathbb{P}^2\times\mathbb{P}^2\subset\mathbb{P}^8$. Alternatively, we can obtain the transformation $\psi:\mathbb{P}^6\dashrightarrow\mathbf{S}\subset\mathbb{P}^8$ by restriction to a general $\mathbb{P}^6\subset\mathbb{P}^8$ of the special Cremona transformation $\mathbb{P}^8\dashrightarrow\mathbb{P}^8$ of type $(2,2)$.

Example 6.9 ($r=2;n=6;a=3;d=2$)**.** See also [38] and [39]. If $X=\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1)\oplus\mathscr{O}(4))$ or $X=\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(2)\oplus\mathscr{O}(3))$, then $|\mathscr{I}_{X,\mathbb{P}^6}(2)|$ defines a birational transformations $\psi:\mathbb{P}^6\dashrightarrow\mathbf{S}\subset\mathbb{P}^9$ of type $(2,2)$ whose base locus is X and whose image is $\mathbf{S}=\mathbb{G}(1,4)$.

Example 6.10 ($r=2,3;n=6,7;a=5;d=1$)**.** See also [42, III Theorem 3.8]. If $X=\mathbb{G}(1,4)\subset\mathbb{P}^9\subset\mathbb{P}^{10}$, then $|\mathscr{I}_{X,\mathbb{P}^{10}}(2)|$ defines a birational transformation $\psi:\mathbb{P}^{10}\dashrightarrow\mathbf{S}\subset\mathbb{P}^{15}$ of type $(2,1)$ whose base locus is X and whose image is the spinorial variety $\mathbf{S}=S^{10}\subset\mathbb{P}^{15}$. Restricting ψ to a general $\mathbb{P}^7\subset\mathbb{P}^{10}$ (resp. $\mathbb{P}^6\subset\mathbb{P}^{10}$) we obtain a special birational transformation $\mathbb{P}^7\dashrightarrow\mathbf{S}\subset\mathbb{P}^{12}$ (resp. $\mathbb{P}^6\dashrightarrow\mathbf{S}\subset\mathbb{P}^{11}$) whose dimension of the base locus is $r=3$ (resp. $r=2$) and whose image is a linear section of $S^{10}\subset\mathbb{P}^{15}$. In the first case $\mathbf{S}=\overline{\psi(\mathbb{P}^7)}$ is smooth while in the second case the singular locus of $\mathbf{S}=\overline{\psi(\mathbb{P}^6)}$ consists of 5 lines, image of the 5 Segre 3-folds containing del Pezzo surface of degree 5 and spanned by its pencils of conics.

Example 6.11 ($r=2,3;n=6,7;a=6;d=1$)**.** See also [38], [39] and [42, III Theorem 3.8]. We have a birational transformation $\psi:\mathbb{P}^8\dashrightarrow\mathbb{G}(1,5)\subset\mathbb{P}^{14}$ of type $(2,1)$ whose base locus is $\mathbb{P}^1\times\mathbb{P}^3\subset\mathbb{P}^7\subset\mathbb{P}^8$ and whose image is $\mathbb{G}(1,5)$. Restricting ψ to a general $\mathbb{P}^7\subset\mathbb{P}^8$ we obtain a birational transformation $\mathbb{P}^7\dashrightarrow\mathbf{S}\subset\mathbb{P}^{13}$ whose base locus X is a rational normal scroll and whose image $\mathbf{S}$ is a smooth linear section of $\mathbb{G}(1,5)\subset\mathbb{P}^{14}$. Restricting ψ to a general $\mathbb{P}^6\subset\mathbb{P}^8$ we obtain a birational transformation $\psi=\psi|_{\mathbb{P}^6}:\mathbb{P}^6\dashrightarrow\mathbf{S}\subset\mathbb{P}^{12}$ whose base locus X is a rational normal scroll (hence either $X=\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1)\oplus\mathscr{O}(3))$ or $X=\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(2)\oplus\mathscr{O}(2))$) and whose image $\mathbf{S}$ is a singular linear section of $\mathbb{G}(1,5)\subset\mathbb{P}^{14}$. In this case, we denote by $Y\subset\mathbf{S}$ the base locus of the inverse of ψ and by $F=(F_0,\ldots,F_5):\mathbb{P}^5\dashrightarrow\mathbb{P}^5$ the restriction of ψ to $\mathbb{P}^5=\mathrm{Sec}(X)$. We

have

$$\begin{aligned} Y &= \overline{\psi(\mathbb{P}^5)} = \overline{F(\mathbb{P}^5)} = \mathbb{G}(1,3) \subset \mathbb{P}^5 \subset \mathbb{P}^{12}, \\ J_4 &:= \left\{x = [x_0,\ldots,x_5] \in \mathbb{P}^5 \setminus X : \operatorname{rank}\left((\partial F_i/\partial x_j(x))_{i,j}\right) \leq 4\right\}_{\mathrm{red}} \\ &= \left\{x = [x_0,\ldots,x_5] \in \mathbb{P}^5 \setminus X : \dim\left(\overline{F^{-1}(F(x))}\right) \geq 2\right\}_{\mathrm{red}} \text{ and } \dim(J_4) = 3, \\ \overline{\psi(J_4)} &= (\operatorname{sing}(\mathbf{S}))_{\mathrm{red}} = \mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(2)) \subset Y. \end{aligned}$$

Example 6.12 $(r=3; n=8; a=0; d=5)$**.** See also [24]. If $\mathscr{X} \subset \mathbb{P}^9$ is a general 3-dimensional linear section of $\mathbb{G}(1,5) \subset \mathbb{P}^{14}$, $p \in \mathscr{X}$ is a general point and $X \subset \mathbb{P}^8$ is the image of $\mathscr{X}$ under the projection from p, then the homogeneous ideal of X is generated by quadrics and $|\mathscr{I}_{X,\mathbb{P}^8}(2)|$ defines a Cremona transformation $\mathbb{P}^8 \dashrightarrow \mathbb{P}^8$ of type $(2,5)$.

Example 6.13 $(r=3; n=8; a=1; d=3)$**.** See also [40, Example 4.5]. If $X \subset \mathbb{P}^8$ is the blow-up of the smooth quadric $Q^3 \subset \mathbb{P}^4$ at 5 general points $p_1,\ldots,p_5$ with $|H_X| = |2H_{Q^3} - p_1 - \cdots - p_5|$, then $|\mathscr{I}_{X,\mathbb{P}^8}(2)|$ defines a birational transformation $\psi : \mathbb{P}^8 \dashrightarrow \mathbf{S} \subset \mathbb{P}^9$ of type $(2,3)$ whose base locus is X and whose image is a cubic hypersurface with singular locus of dimension 3.

Example 6.14 $(r=3; n=8; a=1; d=4\text{ - incomplete})$**.** By [2] (see also [9]) there exists a smooth irreducible nondegenerate linearly normal 3-dimensional variety $X \subset \mathbb{P}^8$ with $h^1(X,\mathscr{O}_X) = 0$, degree $\lambda = 11$, sectional genus $g = 5$, having the structure of a scroll $\mathbb{P}_{\mathbb{F}^1}(\mathscr{E})$ with $c_1(\mathscr{E}) = 3C_0 + 5f$ and $c_2(\mathscr{E}) = 10$ and hence having degrees of the Segre classes $s_1(X) = -85$, $s_2(X) = 386$, $s_3(X) = -1330$. Now, by Fact 1.6, $X \subset \mathbb{P}^8$ is arithmetically Cohen-Macaulay and by Riemann-Roch, denoting with C a general curve section of X, we obtain

$$h^0(\mathbb{P}^8,\mathscr{I}_X(2)) = h^0(\mathbb{P}^6,\mathscr{I}_C(2)) = h^0(\mathbb{P}^6,\mathscr{O}_{\mathbb{P}^6}(2)) - h^0(C,\mathscr{O}_C(2)) = 28 - (2\lambda + 1 - g), \tag{6.1}$$

hence $h^0(\mathbb{P}^8,\mathscr{I}_X(2)) = 10$. If the homogeneous ideal of X is generated by quadratic forms or at least if $X = V(H^0(\mathscr{I}_X(2)))$, the linear system $|\mathscr{I}_X(2)|$ defines a rational map $\psi : \mathbb{P}^8 \dashrightarrow \mathbf{S} = \overline{\psi(\mathbb{P}^8)} \subset \mathbb{P}^9$ whose base locus is X and whose image $\mathbf{S}$ is nondegenerate. Now, by (2.6) we deduce $\deg(\psi)\deg(\mathbf{S}) = 2$, from which $\deg(\psi) = 1$ and $\deg(\mathbf{S}) = 2$.

Example 6.15 $(r=3; n=8; a=1; d=4)$**.** See also [14, §4] and [40, Example 4.4]. If $X \subset \mathbb{P}^8$ is a general linear 3-dimensional section of the spinorial variety $S^{10} \subset \mathbb{P}^{15}$, then $|\mathscr{I}_{X,\mathbb{P}^8}(2)|$ defines a birational transformation $\psi : \mathbb{P}^8 \dashrightarrow \mathbf{S} \subset \mathbb{P}^9$ of type $(2,4)$ whose base locus is X and whose image is a smooth quadric.

Example 6.16 $(r=3; n=8; a=2; d=3)$**.** By [17] (see also [8]) there exists a smooth irreducible nondegenerate linearly normal 3-dimensional variety $X \subset \mathbb{P}^8$ with $h^1(X,\mathscr{O}_X) = 0$, degree $\lambda = 10$, sectional genus $g = 4$, having the structure of a scroll $\mathbb{P}_{Q^2}(\mathscr{E})$ with $c_1(\mathscr{E}) = \mathscr{O}_Q(3,3)$ and $c_2(\mathscr{E}) = 8$ and hence having degrees of the Segre classes $s_1(X) = -76$, $s_2(X) = 340$, $s_3(X) = -1156$. By Fact 1.6, $X \subset \mathbb{P}^8$ is arithmetically Cohen-Macaulay and its homogeneous ideal is generated by quadratic forms. So by (6.1) we have $h^0(\mathbb{P}^8,\mathscr{I}_X(2)) = 11$ and the linear system $|\mathscr{I}_X(2)|$ defines a rational map $\psi : \mathbb{P}^8 \dashrightarrow \mathbf{S} \subset \mathbb{P}^{10}$ whose base locus is X and whose image $\mathbf{S}$ is nondegenerate. By (2.6) it follows that $\deg(\psi)\deg(\mathbf{S}) = 4$ and hence $\deg(\psi) = 1$ and $\deg(\mathbf{S}) = 4$.

Example 6.17 $(r=3; n=8; a=3; d=2,3)$**.** By [16] (see also [8]) there exists a smooth irreducible nondegenerate linearly normal 3-dimensional variety $X \subset \mathbb{P}^8$ with $h^1(X,\mathscr{O}_X) = 0$,

degree $\lambda = 9$, sectional genus $g = 3$, having the structure of a scroll $\mathbb{P}_{\mathbb{P}^2}(\mathscr{E})$ with $c_1(\mathscr{E}) = 4$ and $c_2(\mathscr{E}) = 7$ (resp. of a quadric fibration over $\mathbb{P}^1$) and hence having degrees of the Segre classes $s_1(X) = -67$, $s_2(X) = 294$, $s_3(X) = -984$ (resp. $s_1(X) = -67$, $s_2(X) = 295$, $s_3(X) = -997$). By Fact 1.6, $X \subset \mathbb{P}^8$ is arithmetically Cohen-Macaulay and its homogeneous ideal is generated by quadratic forms. So by (6.1) we have $h^0(\mathbb{P}^8, \mathscr{I}_X(2)) = 12$ and the linear system $|\mathscr{I}_X(2)|$ defines a rational map $\psi : \mathbb{P}^8 \dashrightarrow \mathbf{S} \subset \mathbb{P}^{11}$ whose base locus is X and whose image $\mathbf{S}$ is nondegenerate. By (2.6) it follows that $\deg(\psi)\deg(\mathbf{S}) = 8$ (resp. $\deg(\psi)\deg(\mathbf{S}) = 5$) and in particular $\deg(\psi) \neq 0$ i.e. $\psi : \mathbb{P}^8 \dashrightarrow \mathbf{S}$ is generically quasi-finite. Again by Fact 1.6 and Fact 1.5 it follows that ψ is birational and hence $\deg(\mathbf{S}) = 8$ (resp. $\deg(\mathbf{S}) = 5$).

Example 6.18 $(r = 3; n = 8; a = 4; d = 2)$**.** Consider the composition

$$f : \mathbb{P}^1 \times \mathbb{P}^3 \longrightarrow \mathbb{P}^1 \times Q^3 \subset \mathbb{P}^1 \times \mathbb{P}^4 \longrightarrow \mathbb{P}^9,$$

where the first map is the identity of $\mathbb{P}^1$ multiplied by $[z_0, z_1, z_2, z_3] \mapsto [z_0^2, z_0z_1, z_0z_2, z_0z_3, z_1^2 + z_2^2 + z_3^2]$, and the last map is $([t_0,t_1],[y_0,\dots,y_4]) \mapsto [t_0y_0,\dots,t_0y_4,t_1y_0,\dots,t_1y_4] = [x_0,\dots,x_9]$. In the equations defining $\overline{f(\mathbb{P}^1 \times \mathbb{P}^3)} \subset \mathbb{P}^9$, by replacing x_9 with x_0, we obtain the quadrics:

$$\begin{gathered} -x_0x_3 + x_4x_8,\ -x_0x_2 + x_4x_7,\ x_3x_7 - x_2x_8,\ -x_0x_5 + x_6^2 + x_7^2 + x_8^2,\ -x_0x_1 + x_4x_6, \\ x_3x_6 - x_1x_8,\ x_2x_6 - x_1x_7,\ -x_0^2 + x_1x_6 + x_2x_7 + x_3x_8,\ -x_0^2 + x_4x_5,\ x_3x_5 - x_0x_8, \\ x_2x_5 - x_0x_7,\ x_1x_5 - x_0x_6,\ x_1^2 + x_2^2 + x_3^2 - x_0x_4. \end{gathered} \tag{6.2}$$

Denoting with I the ideal generated by quadrics (6.2) and $X = V(I)$, we have that I is saturated (in particular $I_2 = H^0(\mathscr{I}_{X,\mathbb{P}^8}(2))$) and X is smooth. The linear system $|\mathscr{I}_{X,\mathbb{P}^8}(2)|$ defines a birational transformation $\psi : \mathbb{P}^8 \dashrightarrow \mathbf{S} \subset \mathbb{P}^{12}$ whose base locus is X and whose image is the variety $\mathbf{S}$ with homogeneous ideal generated by:

$$\begin{gathered} y_6y_9 - y_5y_{10} + y_2y_{11},\ y_6y_8 - y_4y_{10} + y_1y_{11},\ y_5y_8 - y_4y_9 + y_0y_{11},\ y_2y_8 - y_1y_9 + y_0y_{10}, \\ y_2y_4 - y_1y_5 + y_0y_6,\ y_2^2 + y_5^2 + y_6^2 + y_7^2 - y_7y_8 + y_0y_9 + y_1y_{10} + y_4y_{11} - y_3y_{12}. \end{gathered} \tag{6.3}$$

We have $\deg(\mathbf{S}) = 10$ and $\dim(\mathrm{sing}(\mathbf{S})) = 3$. The inverse of $\psi : \mathbb{P}^8 \dashrightarrow \mathbf{S}$ is defined by:
(6.4)

$$\begin{gathered} -y_7y_8 + y_0y_9 + y_1y_{10} + y_4y_{11},\ y_0y_5 + y_1y_6 - y_4y_7 - y_{11}y_{12},\ y_0y_2 - y_4y_6 - y_1y_7 - y_{10}y_{12}, \\ -y_1y_2 - y_4y_5 - y_0y_7 - y_9y_{12},\ -y_0^2 - y_1^2 - y_4^2 - y_8y_{12},\ -y_3y_8 - y_9^2 - y_{10}^2 - y_{11}^2, \\ -y_3y_4 - y_5y_9 - y_6y_{10} - y_7y_{11},\ -y_1y_3 - y_2y_9 - y_7y_{10} + y_6y_{11},\ -y_0y_3 - y_7y_9 + y_2y_{10} + y_5y_{11}. \end{gathered}$$

Note that $\mathbf{S} \subset \mathbb{P}^{12}$ is the intersection of a quadric hypersurface in $\mathbb{P}^{12}$ with the cone over $\mathbb{G}(1,4) \subset \mathbb{P}^9 \subset \mathbb{P}^{12}$.

Example 6.19 $(r = 3; n = 8; a = 5$ - with non liftable inverse$)$**.** If $X \subset \mathbb{P}^8$ is the blow-up of $\mathbb{P}^3$ at a point p with $|H_X| = |2H_{\mathbb{P}^3} - p|$, then (modulo a change of coordinates) the homogeneous ideal of X is generated by the quadrics:
(6.5)

$$\begin{gathered} x_6x_7 - x_5x_8,\ x_3x_7 - x_2x_8,\ x_5x_6 - x_4x_8,\ x_2x_6 - x_1x_8,\ x_5^2 - x_4x_7,\ x_3x_5 - x_1x_8,\ x_2x_5 - x_1x_7, \\ x_3x_4 - x_1x_6,\ x_2x_4 - x_1x_5,\ x_2x_3 - x_0x_8,\ x_1x_3 - x_0x_6,\ x_2^2 - x_0x_7,\ x_1x_2 - x_0x_5,\ x_1^2 - x_0x_4. \end{gathered}$$

The linear system $|\mathscr{I}_{X,\mathbb{P}^8}(2)|$ defines a birational transformation $\psi:\mathbb{P}^8 \dashrightarrow \mathbb{P}^{13}$ whose base locus is X and whose image is the variety $\mathbf{S}$ with homogeneous ideal generated by:
(6.6)

$$\begin{gathered}
y_8y_{10}-y_7y_{12}-y_3y_{13}+y_5y_{13},\ y_8y_9+y_6y_{10}-y_7y_{11}-y_3y_{12}+y_1y_{13},\ y_6y_9-y_5y_{11}+y_1y_{12},\\
y_6y_7-y_5y_8-y_4y_{10}+y_2y_{12}-y_0y_{13},\ y_3y_6-y_5y_6+y_1y_8+y_4y_9-y_2y_{11}+y_0y_{12},\\
y_3y_4-y_2y_6+y_0y_8,\ y_3^2y_5-y_3y_5^2+y_1y_3y_7-y_2y_3y_9+y_2y_5y_9-y_0y_7y_9-y_1y_2y_{10}+y_0y_5y_{10}.
\end{gathered}$$

We have $\deg(\mathbf{S})=19$, $\dim(\mathrm{sing}(\mathbf{S}))=4$ and the degrees of Segre classes of X are: $s_1=-49$, $s_2=201$, $s_3=-627$. So, by (2.7), we deduce that the inverse of $\psi:\mathbb{P}^8 \dashrightarrow \mathbf{S}$ is not liftable; however, a representative of the equivalence class of ψ^{-1} is defined by:

$$\begin{gathered}
y_{12}^2-y_{11}y_{13},\ y_8y_{12}-y_6y_{13},\ y_8y_{11}-y_6y_{12},\ -y_6y_{10}+y_7y_{11}+y_3y_{12}-y_5y_{12},\ y_8^2-y_4y_{13},\\
y_6y_8-y_4y_{12},\ y_3y_8-y_2y_{12}+y_0y_{13},\ y_6^2-y_4y_{11},\ y_5y_6-y_1y_8-y_4y_9.
\end{gathered} \tag{6.7}$$

We also point out that $\mathrm{Sec}(X)$ has dimension 6 and degree 6 (against Proposition 1.4).

Example 6.20 ($r=3;n=8;a=6;d=2$)**.** See also [38] and [39]. If $X=\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1)\oplus\mathscr{O}(1)\oplus\mathscr{O}(4))$ or $X=\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1)\oplus\mathscr{O}(2)\oplus\mathscr{O}(3))$ or $X=\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(2)\oplus\mathscr{O}(2)\oplus\mathscr{O}(2))$, then $|\mathscr{I}_{X,\mathbb{P}^8}(2)|$ defines a birational transformation $\mathbb{P}^8 \dashrightarrow \mathbf{S}\subset\mathbb{P}^{14}$ of type $(2,2)$ whose base locus is X and whose image is $\mathbf{S}=\mathbb{G}(1,5)$.

Example 6.21 ($r=3;n=8;a=7;d=1$)**.** See also [11, Example 2.7] and [29]. Let $Z=\{p_1,\ldots,p_8\}\subset\mathbb{P}^2$ be such that no 4 of the p_i are collinear and no 7 of the p_i lie on a conic and consider the scroll $\mathbb{P}_{\mathbb{P}^2}(\mathscr{E})\subset\mathbb{P}^7$ associated to the very ample vector bundle $\mathscr{E}$ of rank 2, given as an extension by the following exact sequence $0\to\mathscr{O}_{\mathbb{P}^2}\to\mathscr{E}\to\mathscr{I}_{Z,\mathbb{P}^2}(4)\to 0$. The homogeneous ideal of $X\subset\mathbb{P}^7$ is generated by 7 quadrics and so the linear system $|\mathscr{I}_{X,\mathbb{P}^8}(2)|$ defines a birational transformation $\psi:\mathbb{P}^8 \dashrightarrow \mathbf{S}\subset\mathbb{P}^{15}$ of type $(2,1)$. Since we have $c_1(X)=12$, $c_2(X)=15$, $c_3(X)=6$, we deduce $s_1(\mathscr{N}_{X,\mathbb{P}^8})=-60$, $s_2(\mathscr{N}_{X,\mathbb{P}^8})=267$, $s_3(\mathscr{N}_{X,\mathbb{P}^8})=-909$, and hence $\deg(\mathbf{S})=29$, by (2.6). The base locus of the inverse of ψ is $\psi(\mathbb{P}^7)\simeq\mathbb{P}^6\subset\mathbf{S}\subset\mathbb{P}^{15}$. We also observe that the restriction of $\psi|_{\mathbb{P}^7}:\mathbb{P}^7 \dashrightarrow \mathbb{P}^6$ to a general hyperplane $H\simeq\mathbb{P}^6\subset\mathbb{P}^7$ gives rise to a transformation as in Example 6.6.

Example 6.22 ($r=3;n=8;a=8,9;d=1$)**.** If $X\subset\mathbb{P}^7\subset\mathbb{P}^8$ is a 3-dimensional Edge variety of degree 7 (resp. degree 6), then $|\mathscr{I}_{X,\mathbb{P}^8}(2)|$ defines a birational transformation $\mathbb{P}^8 \dashrightarrow \mathbf{S}\subset\mathbb{P}^{16}$ (resp. $\mathbb{P}^8 \dashrightarrow \mathbf{S}\subset\mathbb{P}^{17}$) of type $(2,1)$ whose base locus is X and whose degree of the image is $\deg(\mathbf{S})=33$ (resp. $\deg(\mathbf{S})=38$). For memory overflow problems, we were not able to calculate the scheme $\mathrm{sing}(\mathbf{S})$; however, it is easy to obtain that $1\leq\dim(\mathrm{sing}(\mathbf{S}))<\dim(Y)=6$ and $\dim(\mathrm{sing}(Y))=1$, where Y denotes the base locus of the inverse.

Example 6.23 ($r=3;n=8;a=10;d=1$)**.** See also [38], [39] and [42, III Theorem 3.8]. We have a birational transformation $\mathbb{P}^{10} \dashrightarrow \mathbb{G}(1,6)\subset\mathbb{P}^{20}$ of type $(2,1)$ whose base locus is $\mathbb{P}^1\times\mathbb{P}^4\subset\mathbb{P}^9\subset\mathbb{P}^{10}$ and whose image is $\mathbb{G}(1,6)$. Restricting it to a general $\mathbb{P}^8\subset\mathbb{P}^{10}$ we obtain a birational transformation $\psi:\mathbb{P}^8 \dashrightarrow \mathbf{S}\subset\mathbb{P}^{18}$ whose base locus X is a rational normal scroll (hence either $X=\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1)\oplus\mathscr{O}(1)\oplus\mathscr{O}(3))$ or $X=\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1)\oplus\mathscr{O}(2)\oplus\mathscr{O}(2))$) and whose image $\mathbf{S}$ is a linear section of $\mathbb{G}(1,6)\subset\mathbb{P}^{20}$. We denote by $Y\subset\mathbf{S}$ the base locus of the inverse

of ψ and by $F=(F_0,\dots,F_9):\mathbb{P}^7 \dashrightarrow \mathbb{P}^9$ the restriction of ψ to $\mathbb{P}^7=\mathrm{Sec}(X)$. We have

$$\begin{aligned}
Y &= \overline{\psi(\mathbb{P}^7)}=\overline{F(\mathbb{P}^7)}=\mathbb{G}(1,4)\subset\mathbb{P}^9\subset\mathbb{P}^{18},\\
J_6 &:= \left\{x=[x_0,\dots,x_7]\in\mathbb{P}^7\setminus X : \mathrm{rank}\left(\left(\partial F_i/\partial x_j(x)\right)_{i,j}\right)\leq 6\right\}_{\mathrm{red}}\\
&= \left\{x=[x_0,\dots,x_7]\in\mathbb{P}^7\setminus X : \dim\left(\overline{F^{-1}\left(F(x)\right)}\right)\geq 2\right\}_{\mathrm{red}} \text{ and } \dim(J_6)=5,\\
\overline{\psi(J_6)} &= (\mathrm{sing}(\mathbf{S}))_{\mathrm{red}}\subset Y \text{ and } \dim\left(\overline{\psi(J_6)}\right)=3.
\end{aligned}$$

7. Summary results

Theorem 7.1. *Table 1 classifies all special quadratic transformations φ as in §1 and with $r\leq 3$.*

As a consequence, we generalize [40, Corollary 6.8].

Corollary 7.2. *Let $\varphi:\mathbb{P}^n \dashrightarrow \mathbf{S}\subseteq\mathbb{P}^{n+a}$ be as in §1. If φ is of type $(2,3)$ and $\mathbf{S}$ has coindex $c=2$, then $n=8$, $r=3$ and one of the following cases holds:*

- *$\Delta=3$, $a=1$, $\lambda=11$, $g=5$, $\mathfrak{B}$ is the blow-up of Q^3 at 5 points;*
- *$\Delta=4$, $a=2$, $\lambda=10$, $g=4$, $\mathfrak{B}$ is a scroll over Q^2;*
- *$\Delta=5$, $a=3$, $\lambda=9$, $g=3$, $\mathfrak{B}$ is a quadric fibration over $\mathbb{P}^1$.*

Proof. We have that $\mathfrak{B}\subset\mathbb{P}^n$ is a QEL-variety of type $\delta=(r-d-c+2)/d=(r-3)/3$ and $n=((2d-1)r+3d+c-2)/d=(5r+9)/3$. From Divisibility Theorem [37, Theorem 2.8], we deduce $(r,n,\delta)\in\{(3,8,0),(6,13,1),(9,18,2)\}$ and from the classification of CC-manifolds [30, Theorem 2.2], we obtain $(r,n,\delta)=(3,8,0)$. Now we apply the results in §5. □

We can also regard Corollary 7.2 in the same spirit of [40, Theorem 5.1], where we have classified the transformations φ of type $(2,2)$, when $\mathbf{S}$ has coindex 1. Moreover, in the same fashion, one can prove the following:

Proposition 7.3. *Let φ be as in §1 and of type $(2,1)$. If $c=2$, then $r\geq 1$ and $\mathfrak{B}$ is $\mathbb{P}^1\times\mathbb{P}^2\subset\mathbb{P}^5$ or one of its linear sections. If $c=3$, then $r\geq 2$ and $\mathfrak{B}$ is either $\mathbb{P}^1\times\mathbb{P}^3\subset\mathbb{P}^7$ or $\mathbb{G}(1,4)\subset\mathbb{P}^9$ or one of their linear sections. If $c=4$, then $r\geq 3$ and $\mathfrak{B}$ is either an OADP 3-fold in $\mathbb{P}^7$ or $\mathbb{P}^1\times\mathbb{P}^4\subset\mathbb{P}^9$ or one of its hyperplane sections.*

In Table 1 we use the following shortcuts:

- **$\exists^*$:** flags cases for which is known a transformation φ with base locus $\mathfrak{B}$ as required, but we do not know if the image $\mathbf{S}$ satisfies all the assumptions in §1;
- **$\exists^{**}$:** flags cases for which is known that there is a smooth irreducible variety $X\subset\mathbb{P}^n$ such that, if $X=V(H^0(\mathscr{I}_X(2)))$, then the linear system $|\mathscr{I}_X(2)|$ defines a birational transformation $\varphi:\mathbb{P}^n\dashrightarrow\mathbf{S}=\overline{\varphi(\mathbb{P}^n)}\subset\mathbb{P}^{n+a}$ as stated;
- **?:** flags cases for which we do not know if there exists at least an abstract variety $\mathfrak{B}$ having the structure and the invariants required;
- **$\exists$:** flags cases for which everything works fine.

r	n	a	λ	g	Abstract structure of $\mathfrak{B}$	d	Δ	c	Existence	
1	3	1	2	0	$v_2(\mathbb{P}^1) \subset \mathbb{P}^2$	1	2	1	$\exists$	Ex. 6.1
	4	0	5	1	Elliptic curve	3	1	0	$\exists$	Ex. 6.2
	4	1	4	0	$v_4(\mathbb{P}^1) \subset \mathbb{P}^4$	2	2	1	$\exists$	Ex. 6.3
	4	3	3	0	$v_3(\mathbb{P}^1) \subset \mathbb{P}^3$	1	5	2	$\exists$	Ex. 6.5
2	4	1	2	0	$\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$	1	2	1	$\exists$	Ex. 6.1
	5	0	4	0	$v_2(\mathbb{P}^2) \subset \mathbb{P}^5$	2	1	0	$\exists$	Ex. 6.3
	5	3	3	0	Hyperplane section of $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$	1	5	2	$\exists$	Ex. 6.5
	6	0	7	1	Elliptic scroll $\mathbb{P}_C(\mathscr{E})$ with $e(\mathscr{E}) = -1$	4	1	0	$\exists$	Ex. 6.6
	6	0	8	3	Blow-up of $\mathbb{P}^2$ at 8 points $p_1, \ldots, p_8$, $\lvert H_{\mathfrak{B}} \rvert = \lvert 4H_{\mathbb{P}^2} - p_1 - \cdots - p_8 \rvert$	4	1	0	$\exists$	Ex. 6.6
	6	1	7	2	Blow-up of $\mathbb{P}^2$ at 6 points $p_0, \ldots, p_5$, $\lvert H_{\mathfrak{B}} \rvert = \lvert 4H_{\mathbb{P}^2} - 2p_0 - p_1 - \cdots - p_5 \rvert$	3	2	1	$\exists$	Ex. 6.7
	6	2	6	1	Blow-up of $\mathbb{P}^2$ at 3 points p_1, p_2, p_3, $\lvert H_{\mathfrak{B}} \rvert = \lvert 3H_{\mathbb{P}^2} - p_1 - p_2 - p_3 \rvert$	2	4	2	$\exists$	Ex. 6.8
	6	3	5	0	$\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1) \oplus \mathscr{O}(4))$ or $\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(2) \oplus \mathscr{O}(3))$	2	5	2	$\exists$	Ex. 6.9
	6	5	5	1	Blow-up of $\mathbb{P}^2$ at 4 points $p_1 \ldots, p_4$, $\lvert H_{\mathfrak{B}} \rvert = \lvert 3H_{\mathbb{P}^2} - p_1 - \cdots - p_4 \rvert$	1	12	3	$\exists$	Ex. 6.10
	6	6	4	0	$\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1) \oplus \mathscr{O}(3))$ or $\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(2) \oplus \mathscr{O}(2))$	1	14	3	$\exists$	Ex. 6.11
3	5	1	2	0	$Q^3 \subset \mathbb{P}^4$	1	2	1	$\exists$	Ex. 6.1
	6	3	3	0	$\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$	1	5	2	$\exists$	Ex. 6.5
	7	1	6	1	Hyperplane section of $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$	2	2	1	$\exists$	Ex. 6.3
	7	5	5	1	Linear section of $\mathbb{G}(1,4) \subset \mathbb{P}^9$	1	12	3	$\exists$	Ex. 6.10
	7	6	4	0	$\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1) \oplus \mathscr{O}(1) \oplus \mathscr{O}(2))$	1	14	3	$\exists$	Ex. 6.11
	8	0	12	6	Scroll $\mathbb{P}_Y(\mathscr{E})$, Y birat. ruled surface, $K_Y^2 = 5$, $c_2(\mathscr{E}) = 8$, $c_1^2(\mathscr{E}) = 20$	5	1	0	?	
	8	0	13	8	Variety obtained as the projection of a Fano variety X from a point $p \in X$	5	1	0	$\exists$	Ex. 6.12
	8	1	11	5	Blow-up of Q^3 at 5 points $p_1, \ldots, p_5$, $\lvert H_{\mathfrak{B}} \rvert = \lvert 2H_{Q^3} - p_1 - \cdots - p_5 \rvert$	3	3	2	$\exists$	Ex. 6.13
	8	1	11	5	Scroll over $\mathbb{P}_{\mathbb{P}^1}(\mathscr{O} \oplus \mathscr{O}(-1))$	4	2	1	$\exists^{**}$	Ex. 6.14
	8	1	12	7	Linear section of $S^{10} \subset \mathbb{P}^{15}$	4	2	1	$\exists$	Ex. 6.15
	8	2	10	4	Scroll over Q^2	3	4	2	$\exists^*$	Ex. 6.16
	8	3	9	3	Scroll over $\mathbb{P}^2$	2	8	3	$\exists^*$	Ex. 6.17
	8	3	9	3	Quadric fibration over $\mathbb{P}^1$	3	5	2	$\exists^*$	Ex. 6.17
	8	4	8	2	Hyperplane section of $\mathbb{P}^1 \times Q^3$	2	10	3	$\exists^*$	Ex. 6.18
	8	6	6	0	Rational normal scroll	2	14	3	$\exists$	Ex. 6.20
	8	7	8	3	$\mathbb{P}_{\mathbb{P}^2}(\mathscr{E})$, where $0 \to \mathscr{O}_{\mathbb{P}^2} \to$ $\to \mathscr{E} \to \mathscr{I}_{\{p_1,\ldots,p_8\},\mathbb{P}^2}(4) \to 0$	1	29	4	$\exists^*$	Ex. 6.21
	8	8	7	2	Edge variety	1	33	4	$\exists^*$	Ex. 6.22
	8	9	6	1	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$	1	38	4	$\exists^*$	Ex. 6.22
	8	10	5	0	Rational normal scroll	1	42	4	$\exists$	Ex. 6.23

TABLE 1. All transformations φ as in §1 and with $r \leq 3$

8. Towards the case of dimension 4

In this section we treat the case in which $r=4$. However, when $\delta=0$, we are well away from having an exhaustive classification.

Proposition 8.1 follows from [37, Propositions 1.3, 3.4, Corollary 3.2] and [30, Theorem 2.2].

Proposition 8.1. *If $r=4$, then either $n=10$, $d\geq 2$, $\langle\mathfrak{B}\rangle=\mathbb{P}^{10}$, or one of the following cases holds:*

- $n=6$, $d=1$, $\delta=4$, $\mathfrak{B}=Q^4\subset\mathbb{P}^5$ *is a quadric;*
- $n=8$, $d=1$, $\delta=2$, $\mathfrak{B}\subset\mathbb{P}^7$ *is either* $\mathbb{P}^1\times\mathbb{P}^3\subset\mathbb{P}^7$ *or a linear section of* $\mathbb{G}(1,4)\subset\mathbb{P}^9$*;*
- $n=8$, $d=2$, $\delta=2$, $\mathfrak{B}$ *is* $\mathbb{P}^2\times\mathbb{P}^2\subset\mathbb{P}^8$*;*
- $n=9$, $d=1$, $\delta=1$, $\mathfrak{B}$ *is a hyperplane section of* $\mathbb{P}^1\times\mathbb{P}^4\subset\mathbb{P}^9$*;*
- $n=10$, $d=1$, $\delta=0$, $\mathfrak{B}\subset\mathbb{P}^9$ *is an OADP-variety.*

In Proposition 8.2, we more generally assume that the image $\mathbf{S}$ is nondegenerate, normal and linearly normal (not necessarily factorial) and furthermore we do not assume Assumptions 1.2 and 1.3. As noted earlier, we have $P_{\mathfrak{B}}(1)=11$ and $P_{\mathfrak{B}}(2)=55-a$ and hence

$$\begin{aligned}P_{\mathfrak{B}}(t) &= \lambda\binom{t+3}{4}+(1-g)\binom{t+2}{3}+(2g-3\lambda+\chi(\mathcal{O}_{\mathfrak{B}})-a+31)\binom{t+1}{2}\\ &\quad +(-g+2\lambda-2\chi(\mathcal{O}_{\mathfrak{B}})+a-21)t+\chi(\mathcal{O}_{\mathfrak{B}}).\end{aligned}$$

Proposition 8.2. *If $r=4$, $n=10$ and $\langle\mathfrak{B}\rangle=\mathbb{P}^{10}$, then one of the following cases holds:*

- $a=10$, $\lambda=7$, $g=0$, $\chi(\mathcal{O}_{\mathfrak{B}})=1$, $\mathfrak{B}$ *is a rational normal scroll;*
- $a=7$, $\lambda=10$, $g=3$, $\chi(\mathcal{O}_{\mathfrak{B}})=1$, $\mathfrak{B}$ *is either*
  - *a hyperplane section of* $\mathbb{P}^1\times Q^4\subset\mathbb{P}^{11}$ *or*
  - $\mathbb{P}(\mathscr{T}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(1))\subset\mathbb{P}^{10}$*;*
- $a=6$, $\lambda=11$, $g=4$, $\chi(\mathcal{O}_{\mathfrak{B}})=1$, $\mathfrak{B}$ *is a quadric fibration over* $\mathbb{P}^1$*;*
- $a=5$, $\lambda=12$, $g=5$, $\chi(\mathcal{O}_{\mathfrak{B}})=1$, $\mathfrak{B}$ *is one of the following:*
  - $\mathbb{P}^4$ *blown up at 4 points* $p_1\ldots,p_4$ *embedded by* $|2H_{\mathbb{P}^4}-p_1-\cdots-p_4|$,
  - *a scroll over a ruled surface,*
  - *a quadric fibration over* $\mathbb{P}^1$*;*
- $a=4$, $\lambda=14$, $g=8$, $\chi(\mathcal{O}_{\mathfrak{B}})=1$, $\mathfrak{B}$ *is either*
  - *a linear section of* $\mathbb{G}(1,5)\subset\mathbb{P}^{14}$ *or*
  - *the product of* $\mathbb{P}^1$ *with a Fano variety of even index;*
- $a=4$, $\lambda=13$, $g=6$, $\chi(\mathcal{O}_{\mathfrak{B}})=1$, $\mathfrak{B}$ *is either*
  - *a scroll over a birationally ruled surface or*
  - *a quadric fibration over* $\mathbb{P}^1$*;*
- $a=3$, $14\leq\lambda\leq16$, $g\leq11$, $\chi(\mathcal{O}_{\mathfrak{B}})=(-g+2\lambda-18)/3$*;*
- $a=2$, $15\leq\lambda\leq18$, $g\leq14$, $\chi(\mathcal{O}_{\mathfrak{B}})=(-g+2\lambda-19)/3$*;*
- $a=1$, $15\leq\lambda\leq20$, $g\leq17$, $\chi(\mathcal{O}_{\mathfrak{B}})=(-g+2\lambda-20)/3$*;*
- $a=0$, $15\leq\lambda$.

Proof. Denote by $\Lambda\subsetneq C\subsetneq S\subsetneq X\subsetneq\mathfrak{B}$ a sequence of general linear sections of $\mathfrak{B}$ and put $h_\Lambda(2):=h^0(\mathbb{P}^6,\mathcal{O}(2))-h^0(\mathbb{P}^6,\mathscr{I}_\Lambda(2))$. Since C is a nondegenerate curve in $\mathbb{P}^7$, we have $\lambda\geq7$. By Castelnuovo's argument [40, Lemma 6.1], it follows that

$$7\leq\min\{\lambda,13\}\leq h_\Lambda(2)\leq 28-h^0(\mathbb{P}^{10},\mathscr{I}_{\mathfrak{B}}(2))=17-a \tag{8.1}$$

and in particular we have $a\leq10$. Moreover

- if $\lambda \geq 13$, then $h_{\Lambda}(2) \geq 13$ and $a \leq 4$, by (8.1);
- if $\lambda \geq 15$, then $h_{\Lambda}(2) \geq 14$ and $a \leq 3$, by Castelnuovo Lemma [10, Lemma 1.10];
- if $\lambda \geq 17$, then $h_{\Lambda}(2) \geq 15$ and $a \leq 2$, by [10, Theorem 3.1];
- if $\lambda \geq 19$, then $h_{\Lambda}(2) \geq 16$ and $a \leq 1$, by [10, Theorem 3.8];
- if $\lambda \geq 21$, then $h_{\Lambda}(2) \geq 17$ and $a = 0$, by [36, Theorem 2.17(b)].

According to the above statements, we consider the refinement $\theta = \theta(\lambda)$ of Castelnuovo's bound $\rho = \rho(\lambda)$, contained in [10, Theorem 2.5]. So, we have

$$K_{\mathfrak{B}} \cdot H_{\mathfrak{B}}^3 = 2g - 2 - 3\lambda \leq 2\theta(\lambda) - 2 - 3\lambda \leq 2\rho(\lambda) - 2 - 3\lambda. \tag{8.2}$$

Now, if $t \geq 1$, by Kodaira Vanishing Theorem and Serre Duality, it follows that $P_{\mathfrak{B}}(-t) = h^4(\mathfrak{B}, \mathscr{O}_{\mathfrak{B}}(-t)) = h^0(\mathfrak{B}, K_{\mathfrak{B}} + tH_{\mathfrak{B}})$; hence, if $P_{\mathfrak{B}}(-t) \neq 0$, then $K_{\mathfrak{B}} + tH_{\mathfrak{B}}$ is an effective divisor and we have either $K_{\mathfrak{B}} \cdot H_{\mathfrak{B}}^3 > -tH_{\mathfrak{B}}^4 = -t\lambda$ or $K_{\mathfrak{B}} \sim -tH_{\mathfrak{B}}$. Thus, by (8.2) and straightforward calculation, we deduce (see Figure 1):

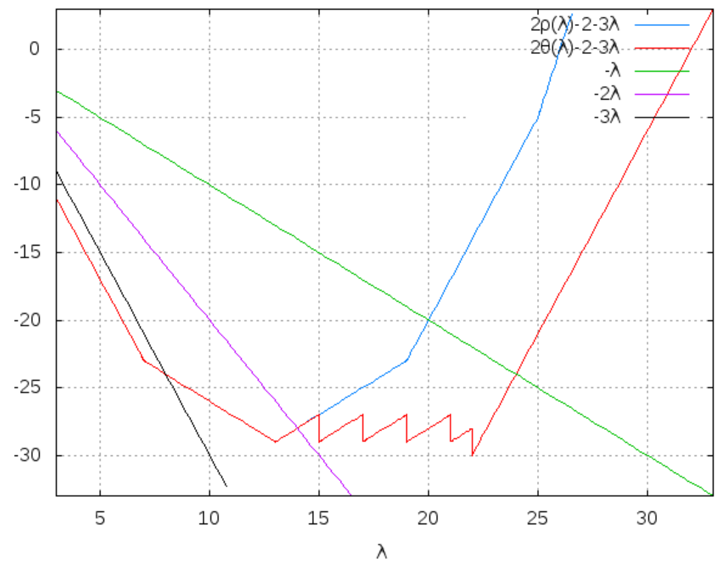

FIGURE 1. Upper bounds of $K_{\mathfrak{B}} \cdot H_{\mathfrak{B}}^3$

(8.2.a) if $\lambda \leq 8$, then either $P_{\mathfrak{B}}(-3) = P_{\mathfrak{B}}(-2) = P_{\mathfrak{B}}(-1) = 0$ or $\lambda = 8$ and $K_{\mathfrak{B}} \sim -3H_{\mathfrak{B}}$;
(8.2.b) if $\lambda \leq 14$, then either $P_{\mathfrak{B}}(-2) = P_{\mathfrak{B}}(-1) = 0$ or $\lambda = 14$ and $K_{\mathfrak{B}} \sim -2H_{\mathfrak{B}}$;
(8.2.c) if $\lambda \leq 24$, then either $P_{\mathfrak{B}}(-1) = 0$ or $\lambda = 24$ and $K_{\mathfrak{B}} \sim -H_{\mathfrak{B}}$.

In the same way, one also sees that $h^4(\mathfrak{B}, \mathscr{O}_{\mathfrak{B}}) = 0$ whenever $\lambda \leq 31$. Now we discuss the cases according to the value of a.

Case 8.2.1 ($9 \leq a \leq 10$). We have $\lambda \leq 8$. From the classification of del Pezzo varieties in [19, I §8], we see that the case $\lambda = 8$ with $K_{\mathfrak{B}} \sim -3H_{\mathfrak{B}}$ is impossible and so we obtain $\lambda = 11 - 2a/5$, $g = 1 - a/10$, by (8.2.a). Hence $a = 10$, $\lambda = 7$, $g = 0$ and $\mathfrak{B}$ is a rational normal scroll.

Case 8.2.2 ($5 \leq a \leq 8$). We have $\lambda \leq 12$. By (8.2.b) we obtain $g = (3\lambda + a - 31)/2$ and $\chi(\mathscr{O}_{\mathfrak{B}}) = (\lambda + a - 11)/6$ and, since $\chi(\mathscr{O}_{\mathfrak{B}}) \in \mathbb{Z}$, we obtain $\lambda = 17 - a$, $g = 10 - a$, $\chi(\mathscr{O}_{\mathfrak{B}}) = 1$. So, we

can determine the abstract structure of $\mathfrak{B}$ by [17], [6], [27, Theorem 2], [7, Lemmas 4.1 and 6.1] and we also deduce that the case $a = 8$ does not occur, by [16].

Case 8.2.3 ($a = 4$). We have $\lambda \leq 14$. Again by (8.2.b), we deduce that either $g = (3\lambda - 27)/2$ and $\chi(\mathscr{O}_{\mathfrak{B}}) = (\lambda - 7)/6$ or $\mathfrak{B}$ is a Mukai variety with $\lambda = 14$ ($g = 8$ and $\chi(\mathscr{O}_{\mathfrak{B}}) = 1$). In the first case, since $\chi(\mathscr{O}_{\mathfrak{B}}) \in \mathbb{Z}$ and $g \geq 0$, we obtain $\lambda = 13$, $g = 6$, $\chi(\mathscr{O}_{\mathfrak{B}}) = 1$ and then we can determine the abstract structure of $\mathfrak{B}$ by [26, Theorem 1] and [7, Lemmas 4.1 and 6.1]. In the second case, if $b_2 = b_2(\mathfrak{B}) = 1$ then $\mathfrak{B}$ is a linear section of $\mathbb{G}(1,5) \subset \mathbb{P}^{14}$, otherwise $\mathfrak{B}$ is a Fano variety of product type, see [34, Theorems 2 and 7].

Case 8.2.4 ($a = 3$). We have $\lambda \leq 16$ and $\chi(\mathscr{O}_{\mathfrak{B}}) = (-g + 2\lambda - 18)/3$, by (8.2.c). Moreover, if $\lambda \leq 14$, by (8.2.b) it follows that $\lambda = 14$, $g = 7$ and $\chi(\mathscr{O}_{\mathfrak{B}}) = 1$.

Case 8.2.5 ($a = 2$). We have $\lambda \leq 18$ and $\chi(\mathscr{O}_{\mathfrak{B}}) = (-g + 2\lambda - 19)/3$, by (8.2.c). Moreover, by (8.2.b) it follows that $\lambda \geq 15$.

Case 8.2.6 ($a = 1$). We have $\lambda \leq 20$ and $\chi(\mathscr{O}_{\mathfrak{B}}) = (-g + 2\lambda - 20)/3$, by (8.2.c). Moreover, if $\lambda \leq 14$, by (8.2.b) it follows that $\lambda = 10$, $g = 0$, $\chi(\mathscr{O}_{\mathfrak{B}}) = 0$, which is of course impossible.

Case 8.2.7 ($a = 0$). If $\lambda \leq 14$, by (8.2.b) and (8.2.c) it follows that $\lambda = 11$, $g = 1$, $\chi(\mathscr{O}_{\mathfrak{B}}) = 0$. Thus, $\mathfrak{B}$ must be an elliptic scroll and φ must be of type $(2,6)$; so, by (8.3) we obtain the contradiction $c_2(\mathfrak{B}) \cdot H_{\mathfrak{B}}^2 = (990 + c_4(\mathfrak{B}))/37 = 990/37 \notin \mathbb{Z}$.

□

Remark 8.3. Under the hypothesis of Proposition 8.2, reasoning as in Proposition 2.2, we obtain that if φ is of type $(2,d)$, then

$$37c_2(\mathfrak{B}) \cdot H_{\mathfrak{B}}^2 - c_4(\mathfrak{B}) = -231\lambda + 188g + (1 - 9d)\Delta + 3396, \tag{8.3}$$

$$37c_3(\mathfrak{B}) \cdot H_{\mathfrak{B}} + 7c_4(\mathfrak{B}) = 655\lambda - 428g + (26d - 7)\Delta - 5716. \tag{8.4}$$

Remark 8.4. If Eisenbud-Green-Harris Conjecture $I_{11,6}$ holds (see [15]), then we have that $\lambda \leq 24$, even in the case with $a = 0$. If $a = 0$ and $\lambda \leq 24$, we have $g \leq \theta(24) = 25$ and one of the following cases holds:

- $\lambda = 24$, $g = 25$, $\chi(\mathscr{O}_{\mathfrak{B}}) = 1$ and $\mathfrak{B}$ is a Fano variety of coindex 4;
- $g \leq 24$ and $\chi(\mathscr{O}_{\mathfrak{B}}) = (-g + 2\lambda - 21)/3$.

Example 8.5. Note that in Proposition 8.1, all cases with $\delta > 0$ really occur (see §6); when $\delta = 0$, an example is obtained by taking a general 4-dimensional linear section of $\mathbb{P}^1 \times \mathbb{P}^5 \subset \mathbb{P}^{11} \subset \mathbb{P}^{12}$. Below we collect some examples of special quadratic birational transformations appearing in Proposition 8.2.

- If $X \subset \mathbb{P}^{10}$ is a (smooth) 4-dimensional rational normal scroll, then $|\mathscr{I}_{X,\mathbb{P}^{10}}(2)|$ defines a birational transformation $\psi : \mathbb{P}^{10} \dashrightarrow \mathbb{G}(1,6) \subset \mathbb{P}^{20}$ of type $(2,2)$.
- If $X \subset \mathbb{P}^{10}$ is a general hyperplane section of $\mathbb{P}^1 \times Q^4 \subset \mathbb{P}^{11}$, then $|\mathscr{I}_{X,\mathbb{P}^{10}}(2)|$ defines a birational transformation $\psi : \mathbb{P}^{10} \dashrightarrow \overline{\psi(\mathbb{P}^{10})} \subset \mathbb{P}^{17}$ of type $(2,2)$ whose image has degree 28.
- If $X = \mathbb{P}(\mathscr{T}_{\mathbb{P}^2} \oplus \mathscr{O}_{\mathbb{P}^2}(1)) \subset \mathbb{P}^{10}$, since $h^1(X, \mathscr{O}_X) = h^1(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}) = 0$, $|\mathscr{I}_{X,\mathbb{P}^{10}}(2)|$ defines a birational transformation $\psi : \mathbb{P}^{10} \dashrightarrow \overline{\psi(\mathbb{P}^{10})} \subset \mathbb{P}^{17}$ (see Facts 1.6 and 1.5).

- There exists a smooth linearly normal 4-dimensional variety $X \subset \mathbb{P}^{10}$ with $h^1(X, \mathscr{O}_X) = 0$, degree 11, sectional genus 4, having the structure of a quadric fibration over $\mathbb{P}^1$ (see [6, Remark 3.2.5]); thus $|\mathscr{I}_{X,\mathbb{P}^{10}}(2)|$ defines a birational transformation $\psi : \mathbb{P}^{10} \dashrightarrow \overline{\psi(\mathbb{P}^{10})} \subset \mathbb{P}^{16}$ (see Facts 1.6 and 1.5).
- If $X \subset \mathbb{P}^{10}$ is the blow-up of $\mathbb{P}^4$ at 4 general points $p_1, \ldots, p_4$, embedded by $|2H_{\mathbb{P}^4} - p_1 - \cdots - p_4|$, then $|\mathscr{I}_{X,\mathbb{P}^{10}}(2)|$ defines a birational transformation $\psi : \mathbb{P}^{10} \dashrightarrow \overline{\psi(\mathbb{P}^{10})} \subset \mathbb{P}^{15}$ whose image has degree 29; in this case $\mathrm{Sec}(X)$ is a complete intersection of two cubics.
- If $X \subset \mathbb{P}^{10}$ is a general 4-dimensional linear section of $\mathbb{G}(1,5) \subset \mathbb{P}^{14}$, then $|\mathscr{I}_{X,\mathbb{P}^{10}}(2)|$ defines a birational transformation $\psi : \mathbb{P}^{10} \dashrightarrow \overline{\psi(\mathbb{P}^{10})} \subset \mathbb{P}^{14}$ of type $(2,2)$ whose image is a complete intersection of quadrics.

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